

A short brute force proof of Jacobi's Identity

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1 Introduction

Jacobi's Identity involves the Poisson bracket which can be written as follows:

$$\{F, G\}_{q,p} = \sum_{j=1}^3 \left(\frac{\partial F}{\partial q^j} \frac{\partial G}{\partial p^j} - \frac{\partial G}{\partial q^j} \frac{\partial F}{\partial p^j} \right) \quad (1)$$

In what follows we omit the subscripts q, p in the Poisson bracket which we will simply write as $\{F, G\}$ with the order of q, p understood.

Jacobi's Identity reads as follows:

$$\{F, \{G, H\}\} + \{G, \{H, F\}\} + \{H, \{F, G\}\} = 0 \quad (2)$$

Note the cyclic symmetry of the identity:

$F \rightarrow G \rightarrow H$ then $G \rightarrow H \rightarrow F$ and, completing the circle, $H \rightarrow F \rightarrow G$.

According to Goldstein [1] "there seems to be no simple way of proving Jacobi's identity for the Poisson bracket without lengthy algebra." There are many proofs of (2), some of which are developed in the context of certain physical problems while others view (2) as a purely mathematical object. Brute force proofs involve a tsunami of symbols and a correspondingly high chance of error. In what follows I have tried to eliminate the sources of error. This is still a "bare hands" proof which does not involve any knowledge beyond the definition and basic calculus.

The Poisson bracket has a basic asymmetry as evidenced by the following relationship which might be thought to assist a proof:

$$\{F, G\} + \{G, F\} = 0 \quad (3)$$

That this is the case follows directly from the definition in (1) ie:

$$\{G, F\} = \sum_{j=1}^3 \left(\frac{\partial G}{\partial q^j} \frac{\partial F}{\partial p^j} - \frac{\partial F}{\partial q^j} \frac{\partial G}{\partial p^j} \right) = -\{F, G\} \quad (4)$$

However, the components of (2) necessarily involve second derivatives with cyclic permutations of F, G and H and it is not immediately obvious that they cancel out to zero. A brute force proof has to deal with all the tedium of keeping track of the order of all the terms but in what follows I have tried to minimise the tedium.

2 The proof

We start with a complete expansion of each component in (2) as follows. This is neither hard nor profound and simply requires keeping track of indices:

$$\begin{aligned} \{F, \{G, H\}\} &= \sum_{i=1}^n \left(\frac{\partial F}{\partial q_i} \frac{\partial \{G, H\}}{\partial p_i} - \frac{\partial \{G, H\}}{\partial q_i} \frac{\partial F}{\partial p_i} \right) \\ &= \sum_{j=1}^n \left\{ \frac{\partial F}{\partial q_i} \frac{\partial}{\partial p_i} \left(\sum_{j=1}^n \left(\frac{\partial G}{\partial q_j} \frac{\partial H}{\partial p_j} - \frac{\partial H}{\partial q_j} \frac{\partial G}{\partial p_j} \right) \right) - \frac{\partial F}{\partial p_i} \frac{\partial}{\partial q_i} \left(\sum_{j=1}^n \left(\frac{\partial G}{\partial q_j} \frac{\partial H}{\partial p_j} - \frac{\partial H}{\partial q_j} \frac{\partial G}{\partial p_j} \right) \right) \right\} \\ &= \sum_{i=1}^n \left\{ \frac{\partial F}{\partial q_i} \left[\underbrace{\sum_{j=1}^n \left[\frac{\partial G}{\partial q_j} \frac{\partial^2 H}{\partial p_i \partial p_j} \right]}_1 + \underbrace{\left[\frac{\partial H}{\partial p_j} \frac{\partial^2 G}{\partial p_i \partial q_j} \right]}_2 - \underbrace{\left[\frac{\partial H}{\partial q_j} \frac{\partial^2 G}{\partial p_i \partial p_j} \right]}_3 - \underbrace{\left[\frac{\partial G}{\partial p_j} \frac{\partial^2 H}{\partial p_i \partial q_j} \right]}_4 \right] \right. \\ &\quad \left. - \frac{\partial F}{\partial p_i} \sum_{j=1}^n \left[\underbrace{\left[\frac{\partial G}{\partial q_j} \frac{\partial^2 H}{\partial q_i \partial p_j} \right]}_6 + \underbrace{\left[\frac{\partial H}{\partial p_j} \frac{\partial^2 G}{\partial q_i \partial q_j} \right]}_{12} - \underbrace{\left[\frac{\partial H}{\partial q_j} \frac{\partial^2 G}{\partial q_i \partial p_j} \right]}_9 - \underbrace{\left[\frac{\partial G}{\partial p_j} \frac{\partial^2 H}{\partial q_i \partial q_j} \right]}_{10} \right] \right\} \end{aligned} \quad (5)$$

Now we can actually simply replace F, G, H in (5) by G, H, F in that order and go straight to the final two lines of (5) being very careful about getting the replacements correct:

$$\begin{aligned}
\{G, \{H, F\}\} &= \sum_{i=1}^n \left\{ \frac{\partial G}{\partial q_i} \left[\sum_{j=1}^n \underbrace{\left[\frac{\partial H}{\partial q_j} \frac{\partial^2 F}{\partial p_i \partial p_j} \right]}_5 + \underbrace{\left[\frac{\partial F}{\partial p_j} \frac{\partial^2 H}{\partial p_i \partial q_j} \right]}_6 - \underbrace{\left[\frac{\partial F}{\partial q_j} \frac{\partial^2 H}{\partial p_i \partial p_j} \right]}_1 - \underbrace{\left[\frac{\partial H}{\partial p_j} \frac{\partial^2 F}{\partial p_i \partial q_j} \right]}_7 \right] \right. \\
&\quad \left. - \frac{\partial G}{\partial p_i} \sum_{j=1}^n \left[\underbrace{\left[\frac{\partial H}{\partial q_j} \frac{\partial^2 F}{\partial q_i \partial p_j} \right]}_8 + \underbrace{\left[\frac{\partial F}{\partial p_j} \frac{\partial^2 H}{\partial q_i \partial q_j} \right]}_{10} - \underbrace{\left[\frac{\partial F}{\partial q_j} \frac{\partial^2 H}{\partial q_i \partial p_j} \right]}_4 - \underbrace{\left[\frac{\partial H}{\partial p_j} \frac{\partial^2 F}{\partial q_i \partial q_j} \right]}_{11} \right] \right\} \quad (6)
\end{aligned}$$

We do the same for $\{H, \{F, G\}\}$:

$$\begin{aligned}
\{H, \{F, G\}\} &= \sum_{i=1}^n \left\{ \frac{\partial H}{\partial q_i} \left[\sum_{j=1}^n \underbrace{\left[\frac{\partial F}{\partial q_j} \frac{\partial^2 G}{\partial p_i \partial p_j} \right]}_3 + \underbrace{\left[\frac{\partial G}{\partial p_j} \frac{\partial^2 F}{\partial p_i \partial q_j} \right]}_8 - \underbrace{\left[\frac{\partial G}{\partial q_j} \frac{\partial^2 F}{\partial p_i \partial p_j} \right]}_5 - \underbrace{\left[\frac{\partial F}{\partial p_j} \frac{\partial^2 G}{\partial p_i \partial q_j} \right]}_9 \right] \right. \\
&\quad \left. - \frac{\partial H}{\partial p_i} \sum_{j=1}^n \left[\underbrace{\left[\frac{\partial F}{\partial q_j} \frac{\partial^2 G}{\partial q_i \partial p_j} \right]}_2 + \underbrace{\left[\frac{\partial G}{\partial p_j} \frac{\partial^2 F}{\partial q_i \partial q_j} \right]}_{11} - \underbrace{\left[\frac{\partial G}{\partial q_j} \frac{\partial^2 F}{\partial q_i \partial p_j} \right]}_7 - \underbrace{\left[\frac{\partial F}{\partial p_j} \frac{\partial^2 G}{\partial q_i \partial q_j} \right]}_{12} \right] \right\} \quad (7)
\end{aligned}$$

Now in (5)-(7) there are 24 components comprising 12 pairs which reflect a certain symmetry. I will list them exhaustively before pointing out how to exploit the symmetry:

For each i in the outer sums in (5)-(7):

1. $\frac{\partial F}{\partial q_i} \sum_{j=1}^n \left(\frac{\partial G}{\partial q_j} \frac{\partial^2 H}{\partial p_i \partial p_j} \right)$ is paired with $\frac{\partial G}{\partial q_i} \sum_{j=1}^n \left(-\frac{\partial F}{\partial q_j} \frac{\partial^2 H}{\partial p_i \partial p_j} \right)$
2. $\frac{\partial F}{\partial q_i} \sum_{j=1}^n \left(\frac{\partial H}{\partial p_j} \frac{\partial^2 G}{\partial p_i \partial q_j} \right)$ is paired with $-\frac{\partial H}{\partial p_i} \sum_{j=1}^n \left(\frac{\partial F}{\partial q_j} \frac{\partial^2 G}{\partial q_i \partial p_j} \right)$
3. $\frac{\partial F}{\partial q_i} \sum_{j=1}^n \left(-\frac{\partial H}{\partial q_j} \frac{\partial^2 G}{\partial p_i \partial p_j} \right)$ is paired with $\frac{\partial H}{\partial q_i} \sum_{j=1}^n \left(\frac{\partial F}{\partial q_j} \frac{\partial^2 G}{\partial p_i \partial p_j} \right)$
4. $\frac{\partial F}{\partial q_i} \sum_{j=1}^n \left(-\frac{\partial G}{\partial p_j} \frac{\partial^2 H}{\partial p_i \partial q_j} \right)$ is paired with $-\frac{\partial G}{\partial p_i} \sum_{j=1}^n \left(-\frac{\partial F}{\partial q_j} \frac{\partial^2 H}{\partial q_i \partial p_j} \right)$
5. $\frac{\partial G}{\partial q_i} \sum_{j=1}^n \left(\frac{\partial H}{\partial q_j} \frac{\partial^2 F}{\partial p_i \partial p_j} \right)$ is paired with $\frac{\partial H}{\partial q_i} \sum_{j=1}^n \left(-\frac{\partial G}{\partial q_j} \frac{\partial^2 F}{\partial p_i \partial p_j} \right)$
6. $\frac{\partial G}{\partial q_i} \sum_{j=1}^n \left(\frac{\partial F}{\partial p_j} \frac{\partial^2 H}{\partial p_i \partial q_j} \right)$ is paired with $-\frac{\partial F}{\partial p_i} \sum_{j=1}^n \left(\frac{\partial G}{\partial q_j} \frac{\partial^2 H}{\partial q_i \partial p_j} \right)$

7. $\frac{\partial G}{\partial q_i} \sum_{j=1}^n \left(-\frac{\partial H}{\partial p_j} \frac{\partial^2 F}{\partial p_i \partial q_j} \right)$ is paired with $-\frac{\partial H}{\partial p_i} \sum_{j=1}^n \left(-\frac{\partial G}{\partial q_j} \frac{\partial^2 F}{\partial q_i \partial p_j} \right)$
8. $\frac{\partial H}{\partial q_i} \sum_{j=1}^n \left(\frac{\partial G}{\partial p_j} \frac{\partial^2 F}{\partial p_i \partial q_j} \right)$ is paired with $-\frac{\partial G}{\partial p_i} \sum_{j=1}^n \left(\frac{\partial H}{\partial q_j} \frac{\partial^2 F}{\partial q_i \partial p_j} \right)$
9. $\frac{\partial H}{\partial q_i} \sum_{j=1}^n \left(-\frac{\partial F}{\partial p_j} \frac{\partial^2 G}{\partial p_i \partial q_j} \right)$ is paired with $-\frac{\partial F}{\partial p_i} \sum_{j=1}^n \left(-\frac{\partial H}{\partial q_j} \frac{\partial^2 G}{\partial q_i \partial p_j} \right)$
10. $-\frac{\partial G}{\partial p_i} \sum_{j=1}^n \left(\frac{\partial F}{\partial p_j} \frac{\partial^2 H}{\partial q_i \partial q_j} \right)$ is paired with $-\frac{\partial F}{\partial p_i} \sum_{j=1}^n \left(-\frac{\partial G}{\partial p_j} \frac{\partial^2 H}{\partial q_i \partial q_j} \right)$
11. $-\frac{\partial G}{\partial p_i} \sum_{j=1}^n \left(-\frac{\partial H}{\partial p_j} \frac{\partial^2 F}{\partial q_i \partial q_j} \right)$ is paired with $-\frac{\partial H}{\partial p_i} \sum_{j=1}^n \left(\frac{\partial G}{\partial p_j} \frac{\partial^2 F}{\partial q_i \partial q_j} \right)$
12. $-\frac{\partial H}{\partial p_i} \sum_{j=1}^n \left(-\frac{\partial F}{\partial p_j} \frac{\partial^2 G}{\partial q_i \partial q_j} \right)$ is paired with $-\frac{\partial F}{\partial p_i} \sum_{j=1}^n \left(\frac{\partial H}{\partial p_j} \frac{\partial^2 G}{\partial q_i \partial q_j} \right)$

Each of the 12 pairs sum to zero over i so that (2), and hence the sum of (5)-(7), sums to zero. To see this we need only observe three things:

2.1 Three principles

1. We assume that F, G, H are such that $\frac{\partial^2 \square}{\partial p \partial q} = \frac{\partial^2 \square}{\partial q \partial p}$ where $\square = F, G$ or H and p, q run over all possibilities for those variables. In physics contexts it will be assumed that the relevant second partials exist and are continuous, hence ensuring that $\frac{\partial^2 \square}{\partial p \partial q} = \frac{\partial^2 \square}{\partial q \partial p}$.
2. A relationship of the form:

$$\sum_{i=1}^n \frac{\partial A}{\partial q_i} \sum_{j=1}^n \frac{\partial B}{\partial p_j} \frac{\partial^2 C}{\partial p_i \partial q_j} = \sum_{i=1}^n \frac{\partial B}{\partial p_i} \sum_{j=1}^n \frac{\partial A}{\partial q_j} \frac{\partial^2 C}{\partial q_i \partial p_j} \quad (8)$$

holds. Thus, for instance:

$\sum_{i=1}^n \frac{\partial F}{\partial q_i} \sum_{j=1}^n \frac{\partial G}{\partial p_j} \frac{\partial^2 H}{\partial p_i \partial q_j} = \sum_{i=1}^n \frac{\partial G}{\partial p_i} \sum_{j=1}^n \frac{\partial F}{\partial q_j} \frac{\partial^2 H}{\partial q_i \partial p_j}$ - see pair 4 above, noting the positive overall sign of the second member of the pair negative sign of the first member of the pair. Assuming the validity of this relationship, then pair 4 becomes zero on summing:

$$\frac{\partial F}{\partial q_i} \sum_{j=1}^n \left(-\frac{\partial G}{\partial p_j} \frac{\partial^2 H}{\partial p_i \partial q_j} \right) = -\frac{\partial F}{\partial q_i} \sum_{j=1}^n \left(\frac{\partial G}{\partial p_j} \frac{\partial^2 H}{\partial p_i \partial q_j} \right) = -\sum_{i=1}^n \frac{\partial G}{\partial p_i} \sum_{j=1}^n \frac{\partial F}{\partial q_j} \frac{\partial^2 H}{\partial q_i \partial p_j} \quad (9)$$

so that:

$$\frac{\partial F}{\partial q_i} \sum_{j=1}^n \left(-\frac{\partial G}{\partial p_j} \frac{\partial^2 H}{\partial p_i \partial q_j} \right) + \sum_{i=1}^n \frac{\partial G}{\partial p_i} \sum_{j=1}^n \frac{\partial F}{\partial q_j} \frac{\partial^2 H}{\partial q_i \partial p_j} = 0 \quad (10)$$

3. Each component of a pair must have opposite parity ie the parity of the first component of pair 4 is negative while the parity of the second component is positive (ie $- \times -$).

If all the 12 pairs conform to the principles mentioned immediately above then the overall sum is zero and (2) is established because the 12 pairs exhaust the universe of possibilities.

To establish the critical relationship (8) we proceed as follows:

$$\begin{aligned} \sum_{i=1}^n \frac{\partial A}{\partial q_i} \sum_{j=1}^n \frac{\partial B}{\partial p_j} \frac{\partial^2 C}{\partial p_i \partial q_j} &= \sum_{i=1}^n \sum_{j=1}^n \frac{\partial A}{\partial q_i} \frac{\partial B}{\partial p_j} \frac{\partial^2 C}{\partial p_i \partial q_j} \\ &= \sum_{i=1}^n \frac{\partial B}{\partial p_i} \sum_{j=1}^n \frac{\partial A}{\partial q_j} \frac{\partial^2 C}{\partial p_j \partial q_i} \\ &= \sum_{i=1}^n \frac{\partial B}{\partial p_i} \sum_{j=1}^n \frac{\partial A}{\partial q_j} \frac{\partial^2 C}{\partial q_i \partial p_j} \end{aligned} \quad (11)$$

It can be seen by inspection that the 12 pairs conform to these 3 principles so that when each pair is summed the result is zero and so (2) is proved.

Of course this proof gives no physical intuition as it is simply based on treating the Poisson bracket as a mathematical object.

Readers wanting a more "sophisticated" proof can consult Landau and Lifshitz [2] where they use the basic differential operators which form the Poisson bracket to show why the cancellations occur.

3 References

1. Herbert. Goldstein, Classical Mechanics (Addison-Wesley, Reading, MA, 1980), 2nd ed., p. 399.

2. L D Landau and E M Lifshitz, Mechanics: Course of Theoretical Physics, Third Edition , Volume 1, Butterworth Heinmann, 2010, pages 136-137

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