A sly application of the Arithmetic Mean - Geometric Mean inequality

Peter Haggstrom
mathsatbondibeach@gmail.com
https://gotohaggstrom.com

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In Chapter 12 of his book "The Cauchy-Schwarz Master Class " [1] J Michael Steele makes the point that the following inequality is a special case of Maclaurin's inequalities ([1],page 179):

$$(xyz)^{\frac{1}{3}} \leq \left(\frac{xy + yz + zx}{3}\right)^{\frac{1}{2}} \leq \frac{x + y + z}{3} \tag{1}$$

He says this is a "sly refinement of the AM-GM inequality" which it is. He does now show how it comes about so here is the proof.

The first point to note that the AM-GM gives us:

$$(xyz)^{\frac{1}{3}} \leq \frac{x + y + z}{3} \tag{2}$$

It is interesting that $\left(\frac{xy+yz+zx}{3}\right)^{\frac{1}{2}}$ can be put between those two quantities.

We assume that $x, y, z \geq 0$. The first part of (1) is proved by a bit of sleight of hand:

$$\begin{aligned}
(xyz)^{\frac{1}{3}} &= [(xy)(yz)(zx)]^{\frac{1}{6}} \\
&= \left([(xy)(yz)(zx)]^{\frac{1}{3}} \right)^{\frac{1}{2}} \\
&\leq \left(\frac{xy + yz + zx}{3}\right)^{\frac{1}{2}}
\end{aligned} \tag{3}$$

To prove the second part we just need to note that since $(x-y)^2 \geq 0$ etc:

$$\begin{aligned}(x-y)^2 =& x^2+y^2-2xy \\ (y-z)^2 =& y^2+z^2-2yz \\ (z-x)^2 =& z^2+x^2-2zx\end{aligned} \tag{4}$$

Hence we have that: $2(x^2+y^2+z^2) \geq 2(xy+yz+zx)$ or $x^2+y^2+z^2 \geq xy+yz+zx$
But:

$$(x+y+z)^2 = x^2+y^2+z^2+2(xy+yz+zx) \tag{5}$$

so

$$(x+y+z)^2 = x^2+y^2+z^2+2(xy+yz+zx) \geq 3(xy+yz+zx) \tag{6}$$

Thus

$$\left(\frac{xy+yz+zx}{3}\right)^{\frac{1}{2}} \leq \left(\frac{(x+y+z)^2}{9}\right)^{\frac{1}{2}} = \frac{x+y+z}{3} \tag{7}$$

1 References

[1] J. Michael Steel, The Cauchy-Schwarz Master Class: An Introduction to the Art of Mathematic
Cambridge University Press, 2004.

2 History

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