Advanced inequality manipulations

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1 Introduction

In this short paper I deal with various inequality problems that are served up to students doing more advanced high school mathematics courses or studying for Olympiad level contests. In general the harder problems require some recognition of how the Cauchy-Schwarz inequality and the Arithmetic Mean - Geometric Mean inequality (AM-GM) can be used in some shape or form. Indeed not a month goes by in the American Mathematical Monthly or one of its related publications without a problem which has a Cauchy-Schwarz or AM-GM dimension to it or convexity properties. Then there are inequality problems based on the Hölder and Minkowski inequalities and much more. For students who want to plumb the depths of the generality of the principles the best book on Cauchy-Schwarz is undoubtedly J Michael Steele, "The Cauchy-Schwarz Master Class", Cambridge University Press, 2004. If this book doesn't get your inequality juices flowing there is no hope! An older book with incredible depth is G Hardy, J E Littlewood and G Polya, "Inequalities", Second Edition, Cambridge University Press, 1952. I find the typesetting in this book hard to read (it was done in the days before Don Knuth's invention of Tex) but it contains a huge amount of material, some of it quite advanced as you would expect given the stature of the three authors.

In what follows I want to explore the ways of solving these inequality problems rather than presenting them as though they come fully formed from deep space. In one sense playing with inequality problems is all about failure - you will go down various dead ends and after much experience you will see what is more likely to work. After a lot of practice this will become intuitive, although there are several well known lines of attack which you can adopt.

2 Background

I am not going to prove the facts below (see Steele's book if you want all the details).

Cauchy's inequality: For real a_i, b_i:

$$a_1\, b_1 + a_2\, b_2 + \cdots + a_n\, b_n = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}\,\sqrt{b_1^2 + b_2^2 + \cdots + b_n^2} \tag{1}$$

Arithmetic-Geometric (AM-GM) inequality: For non-negative real a_i, b_i:

$$(a_1\, a_2 \ldots a_n)^{\frac{1}{n}} \leq \frac{a_1 + a_2 + \cdots + a_n}{n} \tag{2}$$

Jensen's inequality: A function f:[a,b] $\rightarrow$ $\mathbb{R}$ is convex if for all x, $y \in [a, b]$ and all $0 \leq p \leq 1$ one has:

$$f(\, px + (1-p)y\,) \leq p\, f(x) + (1-p)\, f(y) \tag{3}$$

Given the above and non-negative real numbers p_i which satisfy $p_1 + p_2 + \cdots + p_n = 1$ then for all $x_j \in [a, b]$:

$$f\left(\sum_{j=1}^{n} p_j\, x_j\right) \leq p \sum_{j=1}^{n} p_j\, f(x_j) \tag{4}$$

The following problems can be solved in various ways but in what follows we restrict ourselves to the tools discussed above. As you will see, the Cauchy inequality and the AM-GM inequality are extraordinarily powerful.

3 Problem 1

Let $a_1, a_2 \dots a_n$ be positive real numbers. Prove that:

$$\frac{a_1^2}{a_1+a_2} + \frac{a_2^2}{a_2+a_3} + \cdots + \frac{a_{n-1}^2}{a_{n-1}+a_n} + \frac{a_n^2}{a_n+a_1} \geq \frac{a_1+a_2+\cdots+a_n}{2} \tag{5}$$

3.1 Solution to Problem 1

This problem looks like it might submit to a simple approach. If one relabled each of the a_i such that $a_1 \geq a_2 \geq \cdots \geq a_n > 0$ then $a_1 \geq a_2 \implies a_1 + a_2 \leq 2a_1$ and so $\frac{a_1^2}{a_1+a_2} \geq \frac{a_1}{2}$. If you could replicate this logic then you would have that $\frac{a_{i-1}^2}{a_{i-1}+a_i} \geq \frac{a_{i-1}}{2}$ for $1 \leq i \leq n+1$ but this falls down because the final term in the sum is $\frac{a_n^2}{a_n+a_{n+1}}$ rather than $\frac{a_n^2}{a_n+a_1}$. Can we get around that little problem by defining $a_{n+1} = a_1$? If you do that you will get that $a_n + a_{n+1} = a_n + a_1 \leq 2a_1$ and hence $\frac{a_n^2}{a_n+a_1} \geq \frac{a_n^2}{2a_1}$. So this simple approach does not work.

The next line of attack is to rejig the RHS of (5) into something that might involve the Cauchy inequality. Thus:

$$(a_1+a_2+\cdots+a_n)^2 = \Big(\frac{a_1}{\sqrt{a_1+a_2}}\sqrt{a_1+a_2}+\cdots+\frac{a_{n-1}}{\sqrt{a_{n-1}+a_n}}\sqrt{a_{n-1}+a_n}+\frac{a_n}{\sqrt{a_n+a_1}}\sqrt{a_n+a_1}\Big)^2$$

$$\leq \Big(\frac{a_1^2}{a_1+a_2}+\cdots+\frac{a_{n-1}^2}{a_{n-1}+a_n}+\frac{a_n^2}{a_n+a_1}\Big)\big(a_1+a_2+a_2+a_3+\cdots+a_{n-1}+a_n+a_n+a_1\big)$$

$$\text{using the fact that } \Big(\sum_{i=1}^{n} x_i\, y_i\Big)^2 \leq \Big(\sum_{i=1}^{n} x_i^2\Big)\Big(\sum_{i=1}^{n} y_i^2\Big)$$

$$\text{where } x_i = \frac{a_i}{\sqrt{a_i+a_{i+1}}},\ y_i = \sqrt{a_i+a_{i+1}} \text{ and } a_{n+1} = a_1$$

$$= \Big(\frac{a_1^2}{a_1+a_2}+\cdots+\frac{a_{n-1}^2}{a_{n-1}+a_n}+\frac{a_n^2}{a_n+a_1}\Big)\, 2(a_1+a_2+\cdots+a_n)$$

$$\text{Hence: } \frac{a_1^2}{a_1+a_2}+\cdots+\frac{a_{n-1}^2}{a_{n-1}+a_n}+\frac{a_n^2}{a_n+a_1} \geq \frac{a_1+a_2+\cdots+a_n}{2} \tag{6}$$

3.2 Solution 2 to Problem 1

In this solution the following inequality is used for x and y positive:

$$\frac{x^2+y^2}{x+y} \geq \frac{x+y}{2} \tag{7}$$

This comes directly from the fact that $(x-y)^2 \geq 0$:

$$\begin{aligned}
(x-y)^2 &= x^2 - 2xy + y^2 \\
&= 2(x^2+y^2) - (x^2+2xy+y^2) \geq 0 \\
2(x^2+y^2) &\geq (x+y)^2 \\
\frac{x^2+y^2}{x+y} &\geq \frac{x+y}{2}
\end{aligned} \tag{8}$$

As before we define $a_{n+1} = a_1$ and the consider the two sums:

$S = \sum_{i=1}^{n} \frac{a_i^2}{a_i+a_{i+1}}$ and $T = \sum_{i=1}^{n} \frac{a_{i+1}^2}{a_i+a_{i+1}}$. Therefore:

$$S - T = \sum_{i=1}^{n} \frac{(a_i^2 - a_{i+1}^2)}{a_i + a_{i+1}} = \sum_{i=1}^{n} (a_i - a_{i+1}) = 0 \tag{9}$$

Therefore S=T.

$$S + T = \sum_{i=1}^{n} \frac{a_i^2 + a_{i+1}^2}{a_i + a_{i+1}} \geq \sum_{i=1}^{n} \frac{a_i + a_{i+1}}{2} = \sum_{i=1}^{n} a_i \quad \text{using (7)} \tag{10}$$

But since $S = T$ we have that $S \geq \frac{1}{2} \sum_{i=1}^{n} a_i$ which is what we wanted to show.

4 Problem 2

Let $a, b, c > 0$, then:

$$\frac{1}{a(1+b)} + \frac{1}{b(1+c)} + \frac{1}{c(1+a)} \geq \frac{3}{(abc)^{\frac{1}{3}}(1+(abc)^{\frac{1}{3}})} \tag{11}$$

4.1 Solution to Problem 2

This is a hard problem although it looks seductively simple. I will take you through how I tried to solve it so you can see where I hit dead ends and how I got around them.

Remember that the AM-GM inequality says that $\frac{x+y+z}{3} \geq (xyz)^{\frac{1}{3}}$ and it seems obvious to apply this to the LHS of (11). When you do this you get:

$$\frac{1}{a(1+b)} + \frac{1}{b(1+c)} + \frac{1}{c(1+a)} \geq \frac{3}{(abc)^{\frac{1}{3}} \left((1+b)(1+c)(1+a)\right)^{\frac{1}{3}}} \tag{12}$$

Because $\frac{1}{(xyz)^{\frac{1}{3}}} \geq \frac{3}{x+y+z}$ we have:

$$\frac{3}{(abc)^{\frac{1}{3}} \left((1+b)(1+c)(1+a)\right)^{\frac{1}{3}}} \geq \frac{3}{(abc)^{\frac{1}{3}}} \frac{3}{(3+a+b+c)} \tag{13}$$

The problem is that we are now at a dead end because if we try to use the AM-GM inequality again on $\frac{3}{3+a+b+c}$ the inequality goes in the wrong direction and has the wrong form because $\frac{3}{3+a+b+c} \leq \frac{3}{4(3abc)^{\frac{1}{3}}}$. We still want to use the AM-GM inequality but we need to get something that will involve, $a, b, c, (1+a), (1+b),$ and $(1+c)$ in sums of suitable ratios. Thus we need to get something roughly like:

$$\frac{x}{u} + \frac{y}{v} + \frac{z}{w} \tag{14}$$

because with (14) when we apply the AM-GM we will get:

$$\frac{x}{u} + \frac{y}{v} + \frac{z}{w} \geq 3\left(\frac{x}{u}\frac{y}{v}\frac{z}{w}\right)^{\frac{1}{3}} \tag{15}$$

At least that is the general idea and we need to experiment with one of the symmetric factors such as $\frac{1}{a(1+b)}$ to see what might work (bearing in mind we have cyclic symmetry in the factors and that will help cancellations). Thus as a first attempt we try:

$$\frac{1}{a(1+b)} = \frac{1}{1+a+b}\left[\frac{1}{a} + \frac{1}{1+b}\right] \tag{16}$$

Unfortunately this won't work because after applying the AM-GM inequality we will still have a term like $\frac{1}{1+a+b}$ on the bottom which gets us nowhere. If we try:

$$\frac{1+c}{a(1+b)} = \frac{1}{a(1+b)} + \frac{c}{a(1+b)} \tag{17}$$

we again hit a dead end. However, if we form this type of sum there is some hope:

$$1 + \frac{1+abc}{a(1+b)} = \frac{1+a}{a(1+b)} + \frac{b(1+c)}{1+b} \tag{18}$$

Note here that throwing in a term such as abc has some hope of working because of the cyclic symmetry.

As an experiment, if we apply the AM-GM inequality to the RHS of (18) we get:

$$\frac{1+a}{a(1+b)} + \frac{b(1+c)}{1+b} \geq 2\left(\frac{(1+a)}{a(1+b)}\,\frac{b(1+c)}{(1+b)}\right)^{\frac{1}{2}} \tag{19}$$

At this point, given the cyclic symmetry we can begin to see how the ratios might cancel out in a helpful way. So putting it all together we get:

$$1+\frac{1+abc}{a(1+b)}+1+\frac{1+abc}{b(1+c)}+1+\frac{1+abc}{c(1+a)} = \frac{1+a}{a(1+b)}+\frac{b(1+c)}{1+b}+\frac{1+b}{b(1+c)}+\frac{c(1+a)}{1+c}+\frac{1+c}{c(1+a)}+\frac{a(1+b)}{1+a} \tag{20}$$

The LHS of (20) is $3 + (1+abc)S$ where $S = \frac{1}{a(1+b)} + \frac{1}{b(1+c)} + \frac{1}{c(1+a)}$ so we have:

$$3+(1+abc)S=\frac{1+a}{a(1+b)}+\frac{b(1+c)}{1+b}+\frac{1+b}{b(1+c)}+\frac{c(1+a)}{1+c}+\frac{1+c}{c(1+a)}+\frac{a(1+b)}{1+a} \tag{21}$$

By grouping the RHS of (21) as follows the application of the AM-GM inequality becomes useful:

$$\begin{aligned}
&\Big\{\frac{1+a}{a(1+b)}+\frac{1+b}{b(1+c)}+\frac{1+c}{c(1+a)}\Big\}+\Big\{\frac{b(1+c)}{1+b}++\frac{a(1+b)}{1+a}+\frac{c(1+a)}{1+c}\Big\}\\
&\geq 3\Big(\frac{(1+a)}{a(1+b)}\frac{(1+b)}{b(1+c)}\frac{(1+c)}{c(1+a)}\Big)^{\frac{1}{3}}+3\Big(\frac{b(1+c)}{(1+b)}\frac{a(1+b)}{(1+a)}\frac{c(1+a)}{(1+c)}\Big)^{\frac{1}{3}}\\
&\geq 3\Big(\frac{1}{(abc)^{\frac{1}{3}}}+(abc)^{\frac{1}{3}}\Big)\\
&\therefore \frac{(1+abc)S}{3}\geq\frac{1}{(abc)^{\frac{1}{3}}}+(abc)^{\frac{1}{3}}-1 \quad \text{using (21)}
\end{aligned} \tag{22}$$

Let $u=(abc)^{\frac{1}{3}}$, then:

$$S\geq\frac{3}{1+u^3}\Big[\frac{1}{u}+u-1\Big]=\frac{3(1-u+u^2)}{u(1+u)(1-u+u^2)}=\frac{3}{u(1+u)}=\frac{3}{(abc)^{\frac{1}{3}}\Big(1+(abc)^{\frac{1}{3}}\Big)} \tag{23}$$

The main insight in the solution to this problem is the step that culminates in (19) - the addition of 1 transforms the sum into another sum with suitable symmetry for the AM-GM inequality to act upon.

5 Problem 3

Given $a,b,c>0$ show that:

$$\frac{ab}{c}+\frac{bc}{a}+\frac{ca}{b}\geq a+b+c \tag{24}$$

5.1 Solution 1 to Problem 3

The first thing to note is that the AM-GM does not look like it will be much help. Similarly the Cauchy-Schwarz inequality does not look like it has the right structure for this problem. The LHS of (24) is invariant under a cyclic permutation of a, b, c ie $a \to b$, $b \to c$, and $c \to a$. This symmetry suggests that we can assume that $a \geq b \geq c$ without any loss of generality. Thus we have $\frac{1}{c} \geq \frac{1}{b} \geq \frac{1}{a}$ and $ab \geq ca$ and $ac \geq bc$. If we view the LHS of (25) as the product of two increasing sequences $\{\frac{1}{c}, \frac{1}{b}, \frac{1}{a}\}$ and $\{ab, ac, bc\}$ then it would be really nice if we could conclude that an astute re-arrangement of $\{ab, ac, bc\}$ ie $\{ca, bc, ab\}$ would allow us to conclude:

$$\frac{ab}{c} + \frac{bc}{a} + \frac{ca}{b} \geq \frac{1}{c}\,ca + \frac{1}{b}\,bc + \frac{1}{a}\,ab = a + c + b \tag{25}$$

This in fact can done thanks to something known as the "Rearrangement Theorem".

5.1.1 Statement and proof of the Rearrangement Theorem

Let $\{a_i\}_{i=1}^n$ and $\{b_i\}_{i=1}^n$ be sequences of positive numbers increasing or decreasing in the same direction ie $a_1 \geq a_2 \geq \cdots \geq a_n$ and $b_1 \geq b_2 \geq \cdots \geq b_n$ or $a_1 \leq a_2 \leq \cdots \leq a_n$ and $b_1 \leq b_2 \leq \cdots \leq b_n$. Then for any permutation (c_i) of the (b_i) the following inequalities hold:

$$\sum_{i=1}^{n} a_i\, b_i \geq \sum_{i=1}^{n} a_i\, c_i \geq \sum_{i=1}^{n} a_i\, b_{n-i+1} \tag{26}$$

The maximum occurs when the two sequences are ordered the same way while the minimum occurs when they are ordered oppositely. We can actually relax the assumption that the sequences are positive - the result still follows as long as the sequences are similarly ordered.

Proof:

Let $S = a_1b_1 + a_2b_2 + \cdots + a_xb_x + \cdots + a_yb_y + \cdots + a_nb_n$ and $S' = a_1b_1 + a_2b_2 + \cdots + a_xb_y + \cdots + a_yb_x + \cdots + a_nb_n$. In other words, in replace a_xb_x in S by a_xb_y and a_yb_y by a_yb_x. This is just a permutation of the (b_i). You could think of the LHS of (26) as a dot product and visualise the permutation as:

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_x \\ \vdots \\ a_y \\ \vdots \\ a_n \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_y \\ \vdots \\ b_x \\ \vdots \\ b_n \end{pmatrix}$$

It then follows that:

$$S - S^{'} = a_x b_x + a_y b_y - a_x b_y - a_y b_x = (a_x - a_y)(b_x - b_y) \geq 0 \tag{27}$$

since both $a_x - a_y$ and $b_x - b_y$ are either both non-positive or non-negative ie sorted the same way. Therefore, $S \geq S^{'}$ when the sequences are sorted the same. Since our choice of permutation is arbitrary, the result of the theorem follows.

If the sequences are in opposite directions the inequality in (27) is reversed and $S \leq S^{'}$.

5.2 Solution 2 to Problem 3

Because (25) is invariant under a cyclic permutation we can assume without loss of generality that $a \geq b \geq c$ and we see that:

$$\begin{aligned} \frac{a}{c} + \frac{c}{a} \geq 2 \\ \frac{b}{a} + \frac{a}{b} \geq 2 \\ \frac{c}{b} + \frac{b}{c} \geq 2 \end{aligned} \tag{28}$$

That this is the case follows from the use of the AM-GM (or calculus if you are desperate). Thus $\frac{a}{c} + \frac{c}{a} \geq 2\sqrt{\frac{a}{c}\frac{c}{a}} = 2$

Thus:

$$\frac{ab}{c}+\frac{bc}{a}+\frac{ca}{b} \geq a\big(2-\frac{c}{b}\big)+b\big(2-\frac{a}{c}\big)+c\big(2-\frac{b}{a}\big)=2(a+b+c)-\big(\frac{ac}{b}+\frac{ab}{c}+\frac{bc}{a}\big)$$

$$\therefore 2\big(\frac{ab}{c}+\frac{bc}{a}+\frac{ca}{b}\big) \geq 2(a+b+c)$$

$$\text{ie} \quad \frac{ab}{c}+\frac{bc}{a}+\frac{ca}{b} \geq a+b+c \quad (29)$$

6 Chebyshev's inequality

Assume that $\{a_i\}_{i=1}^n$ and $\{b_i\}_{i=1}^n$ are sequences of positive numbers.

If the sequences are ordered the same then:

$$\frac{a_1b_1+a_2b_2+\cdots+a_nb_n}{n} \geq \frac{a_1+a_2+\cdots+a_n}{n}\,\frac{b_1+b_2+\cdots+b_n}{n} \quad (30)$$

If the sequences are oppositely ordered:

$$\frac{a_1b_1+a_2b_2+\cdots+a_nb_n}{n} \leq \frac{a_1+a_2+\cdots+a_n}{n}\,\frac{b_1+b_2+\cdots+b_n}{n} \quad (31)$$

To prove (31) we simply apply the Rearrangement Theorem as follows:

$$\begin{aligned}
a_1b_1+a_2b_2+\cdots+a_nb_n &= a_1b_1+a_2b_2+\cdots+a_nb_n \\
a_1b_1+a_2b_2+\cdots+a_nb_n &\geq a_1b_2+a_2b_3+\cdots+a_nb_1 \\
a_1b_1+a_2b_2+\cdots+a_nb_n &\geq a_1b_3+a_2b_4+\cdots+a_nb_2 \quad (32) \\
\cdots\cdots\cdots\cdots\cdots\cdots&\cdots\cdots\cdots\cdots\cdots\cdots \\
a_1b_1+a_2b_2+\cdots+a_nb_n &\geq a_1b_n+a_2b_1+\cdots+a_nb_{n-1}
\end{aligned}$$

Adding these inequalities we get:

$$n(a_1b_1+a_2b_2+\cdots+a_nb_n) \geq (a_1+a_2+\cdots+a_n)\,(b_1+b_2+\cdots+b_n)$$

$$\text{That is: } \frac{a_1b_1+a_2b_2+\cdots+a_nb_n}{n} \geq \frac{a_1+a_2+\cdots+a_n}{n}\,\frac{b_1+b_2+\cdots+b_n}{n} \quad (33)$$

The form $n\sum a_ib_i \geq \sum a_i \sum b_i$ is often most useful.

Similar reasoning applies to the proof of (32).

7 Problem 4 - Nesbitt's inequality

If $a, b, c > 0$ then:

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2} \tag{34}$$

7.1 Proof 1 of Nesbitt's inequality

The Rearrangement Theorem makes the proof of Nesbitt's inequality straightforward. Cleary (34) is invariant under a cyclic permutation of a, b and c so we may assume that $a \geq b \geq c$. Then $b+c \leq c+a \leq a+b$ so that $\frac{1}{b+c} \geq \frac{1}{c+a} \geq \frac{1}{a+b}$. So we have our two increasing sequences (a, b, c) and $(\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b})$ and the following inequalities hold:

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{b}{b+c} + \frac{c}{c+a} + \frac{a}{a+b} \tag{35}$$

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{c}{b+c} + \frac{a}{c+a} + \frac{b}{a+b} \tag{36}$$

Adding (35) and (36) we have:

$$2\left(\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}\right) \geq \frac{b+c}{b+c} + \frac{c+a}{c+a} + \frac{a+b}{a+b} = 3 \tag{37}$$

Without the Rearrangement Theorem the inequality is harder to prove. One can use Chebyshev's inequality but this is ultimately based upon the Rearrangement Theorem.

7.2 Proof 2 of Nesbitt's inequality

In this proof we use Chebyshev's inequality and also show in detail why the underlying Rearrangement Theorem processes work. As before we can assume that $a \geq b \geq c > 0$ and as a result the following inequalities hold: $\frac{1}{c+b} \geq \frac{1}{a+c} \geq \frac{1}{a+b}$. We thus have two sequences ordered the same: ie (a, b, c) and $(\frac{1}{c+b}, \frac{1}{a+c}, \frac{1}{a+b})$. Let $S = \frac{a}{c+b} + \frac{b}{a+c} + \frac{c}{a+b}$ and so by Chebyshev's inequality:

$$3S \geq (a+b+c)\Big(\frac{1}{c+b} + \frac{1}{a+c} + \frac{1}{a+b}\Big) \geq (a+b+c)\,\frac{3}{\Big((c+b)(a+c)(a+b)\Big)^{\frac{1}{3}}}$$
$$\geq 9\,\frac{a+b+c}{2(a+b+c)} = \frac{9}{2} \quad \text{using the AM-GM twice} \quad (38)$$

Hence $S \geq \frac{3}{2}$ as required.

Note that we use the AM-GM inequality to get $\frac{3}{\Big((c+b)(a+c)(a+b)\Big)^{\frac{1}{3}}}$ and then in the form $\frac{1}{(uvw)^{\frac{1}{3}}} \geq \frac{3}{u+v+w}$ the second time.

To go right back to basics one could set the problem up as follows (essentially replicating the proof of Chebyshev's inequality:

$$S = \frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} \quad (39)$$

$$S \geq \frac{a}{a+c} + \frac{b}{a+b} + \frac{c}{b+c} \quad (40)$$

$$S \geq \frac{a}{a+b} + \frac{b}{b+c} + \frac{c}{a+c} \quad (41)$$

Now in (39) $\frac{c}{a+b} \ngeq \frac{b}{a+b}$ ie the corresponding term in S is not greater than or equal to the corresponding term on the RHS of (40) Similarly in (41) $\frac{c}{a+b} \ngeq \frac{a}{a+b}$. Thus when you compare term by term it looks as though the inequalities don't hold. This is why these inequalities are so subtle - the overall sum does actually dominate appropriately and to see this you need to look at the sign of the difference between LHS and RHS as follows. The sign of the difference in (40) depends on the numerator in:

$$\frac{a-c}{b+c}+\frac{b-a}{c+a}+\frac{c-b}{a+b}=\frac{(a-c)(a+c)(a+b)+(b-a)(b+a)(b+c)+(c-b)(c+b)(a+c)}{(b+c)(c+a)(a+b)}$$

we now ignore the positive denominator so that the sign depends on:

$$(a^2-c^2)(a+b)+(b^2-a^2)(b+c)+(c^2-b^2)(a+c) \geq (a^2-c^2)(b+c)+(b^2-a^2)(b+c)+(c^2-b^2)(b+c)$$
$$= (a^2-c^2+b^2-a^2+c^2-b^2)(b+c) = 0 \quad (42)$$

Noting that $a+b \geq b+c$. Thus the LHS of (40) is greater than or equal to the RHS.

Doing the same for (41) we see that:

$$\frac{a-b}{b+c}+\frac{b-c}{c+a}+\frac{c-a}{a+b}=\frac{(a-b)(a+b)(a+c)+(b-c)(b+c)(a+b)+(c-a)(c+a)(b+c)}{(b+c)(c+a)(a+b)}$$

we now ignore the positive denominator so that the sign depends on:

$$(a^2-b^2)(a+c)+(b^2-c^2)(a+b)+(c^2-a^2)(b+c) \geq (a^2-b^2)(b+c)+(b^2-c^2)(b+c)+(c^2-a^2)(b+c)$$
$$= (a^2-b^2+b^2-c^2+c^2-a^2)(b+c) = 0 \quad (43)$$

Noting that $a+c \geq b+c$ and $a+b \geq b+c$. Thus the LHS of (41) is greater than or equal to the RHS. This shows in detail how the Rearrangement Theorem and Chebyshev's inequality work.

8 Problem 5

Show in two ways that:

$$\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c} \geq 6 \quad (44)$$

where $a, b, c > 0$

8.1 Solution to Problem 5

If we apply the AM-GM inequality to $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$ we see that:

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geq \frac{3}{(abc)^{\frac{1}{3}}} \geq \frac{9}{a+b+c} \quad (45)$$

Therefore on multiplying both sides of (45) by $a+b+c$ we get:

$$1 + \frac{b+c}{a} + 1 + \frac{c+a}{b} + 1 + \frac{a+b}{c} \geq 9 \quad \text{ie} \ \frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c} \geq 6 \quad (46)$$

This demonstrates that with practice you can see ahead so you can work backwards to something useful.

Because (44) is invariant under a cyclic permuation of a, b, c we can suppose that $c \geq b \geq a$ so that $\frac{1}{a} \geq \frac{1}{b} \geq \frac{1}{c}$ and that $b+c \geq a+c \geq a+b$. Thus the two sequences $(b+c, a+c, a+b)$ and $(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are both increasing and the Rearrangement Theorem gives us:

$$\begin{aligned}\frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c} \geq \frac{b+c}{b} + \frac{c+a}{c} + \frac{a+b}{a} &= 1 + \frac{c}{b} + 1 + \frac{a}{c} + 1 + \frac{b}{a} \\ &= 3 + \frac{c}{b} + \frac{a}{c} + \frac{b}{a} \geq 3 + 3\Big(\frac{cab}{bca}\Big)^{\frac{1}{3}} = 6 \quad (47)\end{aligned}$$

History:

15/06/2014 Obvious typo in (1) corrected along with some references to numbered equations.