An application of Bernoulli's inequality to uniform convergence

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1 Background

In one of its forms Bernoulli's inequality says that for $n \in \mathbb{N}$ and $b > 0$:

$$(1+b)^n \geq 1 + nb \tag{1}$$

That this is so follows quickly from the Binomial theorem because for $b > 0$:

$$(1+b)^n = 1 + \binom{n}{1} b + \binom{n}{2} b^2 + \cdots + \binom{n}{n} b^n \geq 1 + nb \tag{2}$$

We can use Bernoulli's inequality to then prove that for $a > 1$:

$$\lim_{n \to \infty} a^{\frac{1}{n}} = 1 \tag{3}$$

We can then show that equation (3) also holds for $0 < a < 1$ and it is that interval we will be interested in below.

Having developed that infrastructure it is then easy to see that $n^{\frac{1}{n}} \to 1$ for instance.

2 Proof of (3)

To prove (3) we let $a^{\frac{1}{n}} = 1 + b_n$ where $b_n > 0$ for all n and we show that $b_n \to 0$. We have that:

$$(a^{\frac{1}{n}})^n = a = (1 + b_n)^n \geq 1 + nb_n \quad (4)$$

Thus $0 < b_n \leq \frac{a-1}{n}$ and hence as $n \to \infty$, $b_n \to 0$ by the "Sandwich Principle".

Now if $0 < a < 1$ it follows that $\frac{1}{a} > 1$ and we can use (3) to get $(\frac{1}{a})^{\frac{1}{n}} \to 1$. But the algebra of limits allows us to conclude that $a^{\frac{1}{n}} \to 1$. Recall here that the algebra of limits states that if $a_n \to a$ and $b_n \to b$ where $b_n \neq 0$ for all $n \in \mathbb{N}$ and $b \neq 0$, then $\frac{a_n}{b_n} \to \frac{a}{b}$. In our case a_n is simply the constant sequence $a_n = 1$ for all n.

3 An alternative proof of (3)

Because we can express $a^{\frac{1}{n}}$ in terms of the exponential as follows where $a > 1$:

$$a^{\frac{1}{n}} = e^{\frac{1}{n} \ln a} \quad (5)$$

we ought to be able to use the properties of the exponential to construct a proof if there is any justice in the mathematical universe.

Thus we have:

$$e^{\frac{1}{n} \ln a} = 1 + \frac{\ln a}{n} + \frac{1}{2!}\Big(\frac{\ln a}{n}\Big)^2 + \cdots + \frac{1}{k!}\Big(\frac{\ln a}{n}\Big)^k + \ldots \quad (6)$$

We first note that $e^{\frac{1}{n} \ln a} > 1$ for all $a > 1$ and all positive integers n. If we let $u = \frac{\ln a}{n}$ we have:

$$e^{\frac{1}{n} \ln a} = 1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \cdots + \frac{u^k}{k!} + \ldots \quad (7)$$

For fixed $a > 1$, $\frac{\ln a}{n}$ converges to 0 , and so for any $\epsilon > 0$ we can find an N such that $u = \frac{\ln a}{n} < \frac{\epsilon}{2}$ for all $n \geq N$. This allows us to bound (7) by a convergent geometric series since ϵ is assumed to be much smaller than 1:

$$
\begin{aligned}
1+u+\frac{u^2}{2!}+\frac{u^3}{3!}+\dots+\frac{u^k}{k!}+\dots <& 1+\frac{\epsilon}{2}+\frac{1}{2!}\Big(\frac{\epsilon}{2}\Big)^2+\frac{1}{3!}\Big(\frac{\epsilon}{2}\Big)^3+\frac{1}{4!}\Big(\frac{\epsilon}{2}\Big)^4+\dots \\
<& 1+\frac{\epsilon}{2}+\Big(\frac{\epsilon}{2}\Big)^2+\Big(\frac{\epsilon}{3}\Big)^3+\dots \\
=& \frac{1}{1-\frac{\epsilon}{2}} \text{ which we note is greater than 1 for any } 0<\epsilon<1
\end{aligned} \tag{8}
$$

We have that $1 < e^{\frac{1}{n}\ln a} < \frac{1}{1-\frac{\epsilon}{2}}$ which holds for all small positive ϵ hence by the "Sandwich Principle" $e^{\frac{1}{n}\ln a} = a^{\frac{1}{n}\ln a} \to 1$ when $a > 1$. If $0 < a < 1$ then $\ln a < 0$ and we can write $e^{\frac{1}{n}\ln a} = e^{-\frac{1}{n}\ln|a|} = \frac{1}{e^{\frac{1}{n}\ln|a|}}$ and using the same algebra of limits arguments as before we use the result just proved to establish that the limit is also 1 when $0 < a < 1$.

Clearly this proof is not as simple as the first one but it does not present any horrors of principle.

4 Using (3) in a uniform convergence argument

In the theory of Bessel functions a standard result is to prove that:

$$
\lim_{n\to-\frac{1}{2}} J_n(r) = \sqrt{\frac{2}{\pi r}}\cos r \tag{9}
$$

where:

$$
J_n(r) = \frac{r^n}{2^n\Gamma(n+\frac{1}{2})\sqrt{\pi}}\int_{-1}^{1} e^{irt}(1-t^2)^{n-\frac{1}{2}}\,dt \tag{10}
$$

To prove (5) one takes the derivative of $r^{-n}J_n(r)$ and apply a limiting process which gives rise to essentially this (ignoring some constants) proposition:

$$
\lim_{\nu\to 0}\int_{-1}^{1} e^{irt}(1-t^2)^{\nu}\,dt = \int_{-1}^{1}\lim_{\nu\to 0}\left(e^{irt}(1-t^2)^{\nu}\right)dt = \int_{-1}^{1} e^{irt}\,dt = 2i\sin r \tag{11}
$$

The validity of (12) depends on the uniform convergence of the functions involved and we now have the tools to quickly establish that fact. I will then comment on the application of the Lebesgue Dominated Convergence Theorem (LDCT) to problems of this type.

Recall that functions $f_n(t)$ converge uniformly to a function $f(t)$ (usually written as $f_n(t) \rightrightarrows f(t)$) on a domain $\mathcal{D}$ if $\forall\epsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall t \in \mathcal{D}$ the following holds:

$$|f_n(t) - f(t)| < \epsilon \quad \forall n > N \tag{12}$$

Uniform convergence is a global property for a set. Since we are concerned with small $\nu > 0$ we can let $\nu = \frac{1}{n}$ so that $\nu \to 0$ as $n \to \infty$, and thus the function we are looking at is:

$$f_n(t) = e^{irt}(1-t^2)^{\frac{1}{n}} \tag{13}$$

and we want to show that this converges uniformly to $f(t) = e^{irt}$.

So for any $\epsilon > 0$ the following estimate holds because we established that for $0 < a < 1$, (3) holds:

$$\begin{aligned}
|f_n(t) - f(t)| &= |e^{irt}(1-t^2)^{\frac{1}{n}} - e^{irt}| \\
&= |e^{irt}|\,|(1-t^2)^{\frac{1}{n}} - 1| \\
&= |(1-t^2)^{\frac{1}{n}} - 1| \\
&< \epsilon \text{ for n sufficiently large}
\end{aligned} \tag{14}$$

Because we have uniform convergence of the $f_n(t)$ we can push the limiting process through the integral in (12).

5 Comments on the LDCT

One frequently sees airy references along the lines that such and such applies because of the LDCT and generally there is no attempt to actually establish that the hypotheses of the LDCT are satisfied. The reason for this is that in many cases those hypotheses will be satisfied. Recall Littlewood's Three Principles ([1], page 191):
(1) Every measurable set is nearly the union of a finite collection of disjoint open intervals.
(2) Every measurable function is nearly continuous.
(3) Every pointwise convergent sequence of functions is nearly uniformly convergent.

Thus a student has a good chance of correctly asserting that something like $f_n(t)$ here is uniformly convergent without ever sullying his or her hands with the actual hypotheses of the LDCT !

The hypotheses of the LDCT (see [1] page 183) are that the $f_n(t)$ are integrable functions that converge almost everywhere to f over a measurable set E. If there is an integrable function g for which $|f_n| \leq g$ almost everywhere in E, then f is integrable and $\int_E f(t)\,dt = \lim_{n\to\infty} \int_E f_n(t)\,dt$. In the case of (14) these conditions are satisfied and the LDCT does apply. It would be bizarre if it didn't! Weierstrass' theorem that if Riemann integrable functions converge uniformly then the integral of the limit is the limit of the integrals is actually a special case of the LDCT as Bressoud points out in [1] at pages 183-4.

6 References

[1] David M Bressoud, " A Radical Approach to Lebesgue's Theory of Integration" , Cambridge Universty Press, 2008.

7 History

Created
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