An interesting integral substitution trick

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1 Introduction

Because I have an interest in mathematics education a video on Youtube piqued my interest (here is the link `https://www.youtube.com/watch?v=BfZObnTIsYk&t=110s`) . The video relates to an integral from an Indian exam (IIT JEE) which was solved by using a trick that appears out of the heavens. When I looked at the problem more closely it was clear that in relation to the particular integral in (1), the trick was unnecessary but what was more interesting was how one could reverse engineer the trick.

$$\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\, dx \tag{1}$$

The answer is $\frac{\pi}{4}$. In took Mathematica 12.3.1 40.4478 seconds to get the answer.

In the video referenced above the trick in (2) is used to evaluate (1) although it is completely unnecessarly to resort to any tricks for (1). In fact one can solve (1) very simply by a straightforward substitution that also gives a hint as to how the trick was thought up in the first place. Here is the trick:

$$\int_a^b \frac{f(x)}{f(a+b-x) + f(x)}\, dx = \frac{b-a}{2} \tag{2}$$

To integrate (1) just use the substitution $x \mapsto \frac{\pi}{2} - x$ and we get:

$$
\begin{aligned}
I &= \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx \\
&= -\int_{\frac{\pi}{2}}^{0} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx \\
&= \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx \\
\therefore I+I &= \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos x}+\sqrt{\sin x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx \\
\implies I &= \frac{1}{2}\int_0^{\frac{\pi}{2}} dx \\
&= \frac{\pi}{4}
\end{aligned} \tag{3}
$$

In fact, by applying an identical approach as followed in (3) one can prove with the simple substitution $x \mapsto \frac{\pi}{2} - x$ that for all real m:

$$
\int_0^{\frac{\pi}{2}} \frac{\sin^m x}{\sin^m x + \cos^m x}\,dx = \frac{\pi}{4} \tag{4}
$$

In all of mathematics there are very specific approaches or "tricks" which people who have systematically mined a field have developed over time and these ultimately get assimilated into the curriculum or testing regimes. For instance, if you did Fourier Theory at Cambridge under Tom Korner you would know from the study of the Fejer kernel that:

$$
\sum_{r=-n}^{n} (n+1-|r|)e^{irx} = \Big(\sum_{k=0}^{n} e^{i(k-\frac{n}{2})x}\Big)^2 \tag{5}
$$

Unless you are "in the zone" such a result seems like magic (it is used to prove the positivity of the Fejer kernel - see [1])

The Indian integration trick mentioned above is actually interesting because a much better question would be to ask a student to reverse engineer the equation . Where on earth did it come from?

The answer is actually very simple - embarrassingly so. It is simply a species of integration substitution. This is how I reverse engineered it.

What is the simplest integral you know? Surely it is this:

$$\int_a^b dx = b - a \tag{6}$$

That is pretty useless but what about this?

$$\int_a^b \frac{f(x)}{f(x)}\,dx = b - a \tag{7}$$

This is a monument to triviality, so what about this next integral?

$$\int_a^b \frac{f(x)}{f(x) + g(x)}\,dx = ? \tag{8}$$

We can't do much with this because we don't have any useful information about $g(x)$ but what if we place some really simple condition on $g(x)$? The simplest non-trivial condition to place on $g(x)$ is to make it look like $f(x)$, say, $g(x) = f(d - x)$. This is just a simple linear argument of $f(x)$. So let's see where this takes is:

$$\int_a^b \frac{f(x)}{f(x) + f(d - x)}\,dx = ? \tag{9}$$

This last integral literally screams substitution but first we need to get some meaningful representation for d. Remember that we are searching for something in the integrand that in the wash comes out as 1 to get back to our simple integral. So let's try this:

$$\frac{f(a)}{f(a) + f(d - a)} + \frac{f(b)}{f(b) + f(d - b)} \tag{10}$$

Now if $f(d - a) = f(b)$ and $f(d - b) = f(a)$ what we get in (8) is:

$$\frac{f(a)}{f(a) + f(b)} + \frac{f(b)}{f(b) + f(a)} = 1 \tag{11}$$

This suggests that $d = a + b$ which clearly works as indicated. Thus we now know that:

$$g(x) = f(a + b - x) \tag{12}$$

In essence what I have done here is not a million miles away from what Planck did when he modified his radiation law as is explained in [2].

Now we can do some simple substitution as follows. Let $x \mapsto a + b - x$ in the following integral:

$$\begin{aligned} I = & \int_a^b \frac{f(x)}{f(x) + f(a+b-x)}\, dx \\ = & -\int_b^a \frac{f(a+b-x)}{f(a+b-x) + f(x)}\, dx \\ = & \int_a^b \Big(\frac{f(a+b-x) + f(x) - f(x)}{f(a+b-x) + f(x)}\Big)\, dx \\ = & \int_a^b dx - \int_a^b \frac{f(x)}{f(x) + f(a+b-x)}\, dx \\ = & (b-a) - I \implies I = \frac{b-a}{2} \end{aligned} \tag{13}$$

So the trick was really only a form of substitution in disguise!

2 Integral substitution and Bessel functions

In the study of Bessel functions there are what at first sight complicated trigonometrical integrals but with a bit of insight they can be usefully massaged without any really hard work. For instance, the integral form of the Bessel function of order n, $J_n(r)$ is:

$$J_n(r) = \frac{1}{2\pi} \int_0^{2\pi} \cos(n\theta - r\sin\theta)\, d\theta \tag{14}$$

It then follows from this by a simple substitution $\theta \mapsto 2\pi - \theta$ that:

$$J_n(r) = \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - r\sin\theta)\,d\theta \tag{15}$$

(15) is proved as follows (note the splitting of the domain of integration):

$$\begin{aligned}
J_n(r) =& \frac{1}{2\pi}\int_0^{2\pi} \cos(n\theta - r\sin\theta)\,d\theta \\
=& \frac{1}{2\pi}\int_0^{\pi} \cos(n\theta - r\sin\theta)\,d\theta + \frac{1}{2\pi}\int_{\pi}^{2\pi} \cos(n\theta - r\sin\theta)\,d\theta \\
=& \frac{1}{2\pi}\int_0^{\pi} \cos(n\theta - r\sin\theta)\,d\theta - \frac{1}{2\pi}\int_{\pi}^{0} \cos(2n\pi - n\theta) - r\sin(2\pi - \theta))\,d\theta \\
=& \frac{1}{2\pi}\int_0^{\pi} \cos(n\theta - r\sin\theta)\,d\theta + \frac{1}{2\pi}\int_{2\pi}^{0} \cos(n\theta - r\sin\theta)\,d\theta \\
=& \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - r\sin\theta)\,d\theta
\end{aligned} \tag{16}$$

(15) also leads to the following relationships:

$$J_n(r) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \sin n\theta\,\sin(r\sin\theta)\,d\theta \qquad \text{n odd} \tag{17}$$

and

$$J_n(r) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cos n\theta\,\cos(r\sin\theta)\,d\theta \qquad \text{n even} \tag{18}$$

Before proving (17)-(18) we need some other important properties.

A function $f(x)$ is called 2π periodic if:

$$f(x \pm 2\pi) = f(x) \tag{19}$$

Note that (19) implies that $f(x \pm 2k\pi) = f(x)$ for integral k.

On noting that the integrand in (14) and (15) is 2π periodic we can say that:

$$J_n(r) = \frac{1}{2\pi}\int_{\alpha}^{2\pi+\alpha} \cos(n\theta - r\sin\theta)\,d\theta \tag{20}$$

for any angle α.

Some other useful facts about 2π periodic functions (assumed to be integrable on any finite interval) are, for $a, b \in \mathbb{R}$:

$$\int_a^b f(x)\,dx = \int_{a+2\pi}^{b+2\pi} f(x)\,dx = \int_{a-2\pi}^{b-2\pi} f(x)\,dx \tag{21}$$

and

$$\int_{-\pi}^{\pi} f(x+a)\,dx = \int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi+a}^{\pi+a} f(x)\,dx \tag{22}$$

(21) and (22) are in fact exercises from [3] page 58.

(21) is proved by the substitution $x \mapsto x \pm 2\pi$.

To prove $\int_{-\pi}^{\pi} f(x+a)\,dx = \int_{-\pi+a}^{\pi+a} f(x)\,dx$ make the substitution $x \mapsto x+a$ and to prove the remainder we note that:

$$\int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi+a}^{\pi} f(x)\,dx + \int_{-\pi}^{-\pi+a} f(x)\,dx \tag{23}$$

Now make the substitution $x \mapsto x + 2\pi$ in the second integral in (23). Hence,

$$\begin{aligned}
\int_{-\pi}^{\pi} f(x)\,dx &= \int_{-\pi+a}^{\pi} f(x)\,dx + \int_{\pi}^{\pi+a} f(x+2\pi)\,dx \\
&= \int_{-\pi+a}^{\pi} f(x)\,dx + \int_{\pi}^{\pi+a} f(x)\,dx \\
&= \int_{-\pi+a}^{\pi+a} f(x)\,dx \\
&= \int_{-\pi}^{\pi} f(x+a)\,dx
\end{aligned} \tag{24}$$

Now with that behind us to prove (17)-(18) essentially all we need to do is make the substitution $x \mapsto \pi - \theta$:

$$
\begin{aligned}
J_n(r) =& \frac{1}{\pi}\int_0^\pi \cos(n\theta - r\sin\theta)\,d\theta \\
=& \frac{1}{\pi}\int_0^\pi \cos(n\theta)\cos(r\sin\theta)\,d\theta + \frac{1}{\pi}\int_0^\pi \sin n\theta \sin(r\sin\theta)\,d\theta \\
=& I_1 + I_2
\end{aligned}
\tag{25}
$$

Dealing with I_1 and I_2 separately with the substitution $x \mapsto \pi - \theta$:

$$
\begin{aligned}
I_1 =& -\frac{1}{\pi}\int_\pi^0 \cos(n\pi - n\theta)\ \cos(r\sin(\pi-\theta))\,d\theta \\
=& \frac{1}{\pi}\int_0^\pi \cos n\pi\ \cos n\theta\ \cos(r\sin\theta)\,d\theta \\
=& (-1)^n\frac{1}{\pi}\int_0^\pi \cos n\theta\ \cos(r\sin\theta)\,d\theta
\end{aligned}
\tag{26}
$$

Hence $I_1 = (-1)^n I_1$ so that for n odd it follows that $I_1 = 0$.

Imagine if you sat an exam in abstract: Prove $\frac{1}{\pi}\int_0^\pi \cos n\theta\ \cos(r\sin\theta)\,d\theta = 0$.

Note that $f(\theta) = \cos n\theta\ \cos(r\sin\theta) = f(-\theta)$ ie f is an even function.

In these circumstances for n even:

$$
\int_0^\pi f\theta)\,d\theta = \int_0^{\pi/2} f(\theta)\,d\theta + \int_{\pi/2}^\pi f(\theta)\,d\theta = 2\int_0^{\pi/2} f(\theta)\,d\theta
\tag{27}
$$

In the second integral in (27) make the substitution $\theta \mapsto \pi - \theta$ and note that:

$$
\begin{aligned}
f(\pi-\theta) =& \cos(n\pi - n\theta)\cos(r\sin(\pi-\theta)) \\
=& (-1)^n \cos n\theta\ \cos(r\sin\theta) \\
=& \cos n\theta \cos(r\sin\theta) \qquad \text{if n is even} \\
=& f(\theta)
\end{aligned}
\tag{28}
$$

Hence:

$$
\begin{aligned}
\int_0^{\pi} f(\theta)\, d\theta =& \int_0^{\pi/2} f(\theta)\, d\theta - \int_{\pi/2}^{0} f(\pi-\theta)\, d\theta \\
=& \int_0^{\pi/2} f(\theta)\, d\theta + \int_0^{\pi/2} f(\theta)\, d\theta \\
=& 2\int_0^{\pi/2} f(\theta)\, d\theta \\
=& \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cos n\theta\, \cos(r\sin\theta)\, d\theta
\end{aligned} \tag{29}
$$

So for n even we get (18).

Now for I_2 we similarly make the substitution $\theta \mapsto \pi - \theta$ and we get:

$$
\begin{aligned}
I_2 =& \frac{1}{\pi}\int_0^{\pi} \sin n\theta \sin(r\sin\theta)\, d\theta \\
=& -\frac{1}{\pi}\int_{\pi}^{0} \sin(n\pi - n\theta)\sin(r\sin(\pi-\theta))\, d\theta \\
=& \frac{1}{\pi}\int_0^{\pi} -\cos n\pi\, \sin n\theta\, \sin(r\sin\theta)\, d\theta \\
=& \frac{(-1)^{n+1}}{\pi}\int_0^{\pi} \sin n\theta\, \sin(r\sin\theta)\, d\theta \\
=& (-1)^{n+1} I_2
\end{aligned} \tag{30}
$$

So if n is even $I_2 = 0$. We can replicate (27)-(29) with $g(\theta) = \sin n\theta \sin(r\sin\theta)$ which is an even function:

$$
\begin{aligned}
g(\pi-\theta) =& \sin(n\pi - n\theta)\sin(r\sin(\pi-\theta)) \\
=& (-1)^{n+1}\sin n\theta\, \sin(r\sin\theta) \\
=& (-1)^{n+1} g(\theta)
\end{aligned} \tag{31}
$$

Hence making the substitution $\theta \mapsto \pi - \theta$ in the second integral below:

$$\begin{aligned}I_2 =& \frac{1}{\pi}\int_0^{\pi} \sin n\theta \sin(r\sin\theta)\, d\theta \\ =& \frac{1}{\pi}\int_0^{\frac{\pi}{2}} g(\theta)\, d\theta + \frac{1}{\pi}\int_{\frac{\pi}{2}}^{\pi} g(\theta)\, d\theta \\ =& \frac{1}{\pi}\int_0^{\frac{\pi}{2}} g(\theta)\, d\theta - \frac{1}{\pi}\int_{\frac{\pi}{2}}^{0} g(\pi-\theta)\, d\theta \\ =& \frac{1}{\pi}\int_0^{\frac{\pi}{2}} g(\theta)\, d\theta + \frac{1}{\pi}\int_0^{\frac{\pi}{2}} (-1)^{n+1} g(\theta)\, d\theta\end{aligned} \tag{32}$$

So if n is even from the last line of (32) $I_2 = 0$ but if n is odd then $I_2 = \frac{2}{\pi}\int_0^{\pi} \sin n\theta \sin(r\sin\theta)\, d\theta$ which is (17). For much more on Bessel functions the 'bible' is Watson [4].

3 References

[1] Peter Haggstrom, "The nitty gritty of the Fejer kernel", https://gotohaggstrom.com/The%20nitty%20gritty%20of%20Fejers%20Theorem.pdf

[2] Peter Haggstrom, "Deriving the Stefan-Boltzmann law from Planck's law", https://gotohaggstrom.com/Deriving%20the%20Stefan-Boltzmann%20law%20from%20Plancks%20law.pdf

[3] Elias M Stein and Rami Shakarchi, Fourier Analysis: An introduction, Princeton University Press, 2003

[4] G N Watson, A treatise on the Theory of Bessel Functions,Cambridge University Press, Second Edition, 1966.

4 History

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