Area of a circle - limit of triangles

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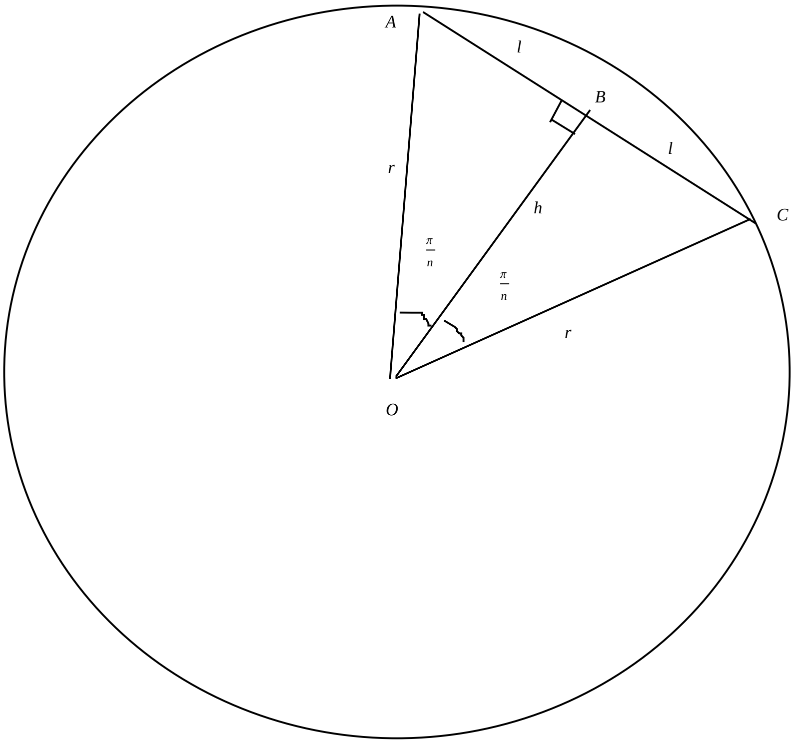
1 Background

As a simple high school exercsise, if you want to prove that the area of a circle of radius r is πr^2 you can approximate the area by n triangles with angle $\frac{2\pi}{n}$ and then let $n \to \infty$. This simple approach to approximation uses a technique which is applicable in several other contexts.

2 How the approximation works

The diagram below gives you the set up:


A
l
B
l
r
h
C
π/n
π/n
r
O

The area of triangle OAB is simply $\frac{1}{2} \times l \times r$ and so the area of the double triangle OAC is $l \times r$. Because $angle \angle OBA$ is a right angle, $l = r\ \sin(\frac{\pi}{n})$ and $h = r\ \cos(\frac{\pi}{n})$.

The area of triangle OAC is thus $r\ \sin(\frac{\pi}{n})\, r\ \cos(\frac{\pi}{n}) = \frac{r^2}{2}\ \sin(\frac{2\pi}{n})$

The total circular angle of 2π is thus divided into n equal triangles of area $\frac{r^2}{2}\ \sin(\frac{2\pi}{n})$ so that the area of the circle is approximated by the limit of this sum as $n \to \infty$:

$$\text{Area of circle} = \lim_{n\to\infty} n\, \frac{r^2}{2}\ \sin(\frac{2\pi}{n}) = \lim_{n\to\infty} \pi r^2\, \frac{\sin(\frac{2\pi}{n})}{\frac{2\pi}{n}} = \pi r^2 \tag{1}$$

since $\frac{\sin(\frac{2\pi}{n})}{\frac{2\pi}{n}} \to 1$ as n $\to \infty$

There are other ways of getting the same result. For instance, consider an elementary sector with small angle $d\theta$. The area of this sector is approximated by $\frac{1}{2}r^2\ \sin(d\theta)$ and because $d\theta$ is small we can approximate $\sin(d\theta)$ by $d\theta$ (this is because $\sin x \approx x$ for small x.) The area of the circle is then $\int_0^{2\pi} \frac{1}{2} r^2\, d\theta = \pi\, r^2$