Background to proving Chebyshev's sum inequality

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1 Introduction

Adam Besenyei's recent article [2] in the Mathematics Magazine provides a great deal of interesting history to Emile Picard's proof of Chebyshev's sum inequality. Picard's proof relied upon translating the problem into a physical setting through the concept of centre of gravity.

Charles Hermite, the French mathematician referred to the Picard proof in his Analysis course notes taught at the Sorbonne in 1881-1882 and you can read the notes in French for yourself and admire the beautiful script [1, pp. 48-49].

My purpose in this short article is to simply derive one important formula mentioned by Besenyei which he did not prove but which can be used to prove Chebyshev's sum inequality. Chebyshev's inequality arises in many areas of mathematics and is especially loved by those setting problems so it is useful to appreciate all of its subtleties.

Apart from Besenyei's article if you want to know more about the relationship between physical analogues and mathematical proofs there is no better book than Mark Levi's "Mathematical Mechanic" [4]. You can read my Amazon review to get a flavour of the scope of the book [5].

2 Chebyshev's sum inequality

If $u_1 \leq u_2 \leq \cdots \leq u_n$ and $v_1 \leq v_2 \leq \cdots \leq v_n$ (or both sequences are weakly decreasing), then:

$$(u_1 + u_2 + \cdots + u_n)(v_1 + v_2 + \cdots + v_n) \leq n(u_1v_1 + u_2v_2 + \cdots + u_nv_n) \tag{1}$$

The integral version of the inequality is:

$$\left(\int_0^1 u \, dx \right) \cdot \left(\int_0^1 v \, dx \right) \leq \int_0^1 uv dx \tag{2}$$

For Chebyshev's generalisation of (2) see [3]

Picard's approach also applies to Chebyshev's rearrangment inequality:

Assume that $u_1 \leq u_2 \leq \cdots \leq u_n$ and $v_1 \leq v_2 \leq \cdots \leq v_n$. If $\tilde{v_1}, \tilde{v_2}, \ldots \tilde{v_n}$ is any permutation of the numbers $v_1, v_2, \ldots v_n$ then:

$$u_1\tilde{v_1} + u_2\tilde{v_2} + \cdots + u_n\tilde{v_n} \leq u_1 v_1 + u_2 v_2 + \cdots + u_n v_n \tag{3}$$

3 An integral identity wrongly attributed

According to Besenyei, Hermite attributed the following integral identity to Fabian Franklin who published it in 1885 but the Russian mathematician Konstantin Alekseevich Andreev had already established it in 1883. Besenyei does not prove this identity since its proof is straightforward:

$$\frac{1}{2}\int_a^b\int_a^b \Big[\big(f(x)-f(y)\big)\big(g(x)-g(y)\big)\Big]\,dxdy = (b-a)\int_a^b f(x)g(x)\,dx - \left(\int_a^b f(x)\,dx\right)\cdot\left(\int_a^b g(x)\,dx\right) \tag{4}$$

To prove this expand the LHS:

$$\frac{1}{2}\int_a^b\int_a^b \Big[\big(f(x)-f(y)\big)\big(g(x)-g(y)\big)\Big]\,dxdy = \frac{1}{2}\int_a^b\int_a^b \Big[\underbrace{f(x)g(x)}_{1} - \underbrace{f(x)g(y)}_{2} - \underbrace{f(y)g(x)}_{3} + \underbrace{f(y)g(y)}_{4}\Big]\,dxdy \tag{5}$$

For component (1) in equation (5) we see that:

$$\frac{1}{2}\int_a^b\int_a^b f(x)g(x)\,dxdy = \frac{1}{2}\int_a^b f(x)g(x)\,dx\int_a^b dy = \frac{b-a}{2}\int_a^b f(x)g(x)\,dx \tag{6}$$

Component (4) is treated similarly (y is just a dummy variable so it can be changed to x) and when added with (6) we get the first term in (4).

For component (2) we have:

$$\frac{1}{2}\int_a^b\int_a^b f(x)g(y)\,dxdy = \frac{1}{2}\int_a^b f(x)\,dx\int_a^b g(y)\,dy = \frac{1}{2}\int_a^b f(x)\,dx\int_a^b g(x)\,dx \tag{7}$$

Similarly for component (3):

$$\frac{1}{2}\int_a^b\int_a^b f(y)g(x)\,dxdy = \frac{1}{2}\int_a^b f(y)\,dy\int_a^b g(x)\,dx = \frac{1}{2}\int_a^b f(x)\,dx\int_a^b g(x)\,dx \tag{8}$$

Adding (7) and (8) we get (4)

4 A discrete analogue of the continuous formula

Besenyei points out the the following identity due to the Russian mathematician Aleksandr Nikolayevich Korkin, a former student of Chebyshev, is the discrete analogue of (4):

$$\frac{1}{n}\sum_{i=1}^{n} x_i y_i = \left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)\left(\frac{1}{n}\sum_{i=1}^{n} y_i\right) + \frac{1}{n^2}\sum_{i<j}(x_i - x_j)(y_i - y_j) \tag{9}$$

This identity can be used to prove Chebyshev's sum inequality but how do you prove it?

The existence of the two means on the RHS of (9) suggests that we introduce means into the LHS. After all, any value x_i can be expressed as some deviation from the mean. Thus we write for all i:

$$x_i = \bar{x} - \delta_i \tag{10}$$

and

$$y_i = \bar{y} - \epsilon_i \tag{11}$$

where $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ and $\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$.

Now from (10) we have:

$$\begin{aligned} -\delta_i =& x_i - \bar{x} \\ =& \frac{nx_i - (x_1 + x_2 + \cdots + x_i + \cdots + x_n)}{n} \\ =& \frac{(n-1)x_i - \sum_{j\neq i}^{n} x_j}{n} \end{aligned} \tag{12}$$

The symbol $\sum_{j\neq i}^{n} x_j$ is intended to mean that all terms except the one for which $j = i$ are taken into account.

Similarly we have:

$$-\epsilon_i = \frac{(n-1)y_i - \sum_{j\neq i}^{n} y_j}{n} \tag{13}$$

Substituting (10) in the LHS of (9) we have:

$$\frac{1}{n}\sum_{i=1}^{n} x_i y_i = \frac{1}{n}\sum_{i=1}^{n}(\bar{x} - \delta_i) y_i \tag{14}$$

Similarly substituting (11) in the LHS of (9) we have:

$$\frac{1}{n}\sum_{i=1}^{n} x_i y_i = \frac{1}{n}\sum_{i=1}^{n}(\bar{y} - \epsilon_i) x_i \tag{15}$$

Adding (14) and (15) and using (12) and (13)we have:

$$\begin{aligned}
\frac{1}{n}\sum_{i=1}^{n} x_i y_i =& \frac{1}{2n}\left(\sum_{i=1}^{n}(\bar{x}-\delta_i)y_i + \sum_{i=1}^{n}(\bar{y}-\epsilon_i)x_i\right)\\
=& \frac{1}{2n}\sum_{i=1}^{n}\left(\bar{x}y_i+\bar{y}x_i\right) - \frac{1}{2n}\sum_{i=1}^{n}\left(\epsilon_i x_i + \delta_i y_i\right)\\
=& \bar{x}\bar{y} + \frac{1}{2n}\sum_{i=1}^{n}\left[x_i\left(\frac{(n-1)y_i - \sum_{j\neq i}^{n} y_j}{n}\right) + y_i\left(\frac{(n-1)x_i - \sum_{j\neq i}^{n} x_j}{n}\right)\right]\\
=& \bar{x}\bar{y} + \frac{1}{2n^2}\sum_{i=1}^{n}\left(2(n-1)x_i y_i - x_i\sum_{j\neq i}^{n} y_j - y_i\sum_{j\neq i}^{n} x_j\right)\\
=& \bar{x}\bar{y} + \frac{1}{2n^2}\sum_{i=1}^{n}\left(2(n-1)x_i y_i - x_i\Big(\sum_{j=1}^{n} y_j - y_i\Big) - y_i\Big(\sum_{j=1}^{n} x_j - x_i\Big)\right)\\
=& \bar{x}\bar{y} + \frac{1}{2n^2}\sum_{i=1}^{n}\left(2n x_i y_i - x_i\sum_{j=1}^{n} y_j - y_i\sum_{j=1}^{n} x_j\right)
\end{aligned} \tag{16}$$

At this stage it may look improbable that this gets us where we want since we need this equality:

$$\frac{1}{2}\sum_{i=1}^{n}\left(2n x_i y_i - x_i\sum_{j=1}^{n} y_j - y_i\sum_{j=1}^{n} x_j\right) = \sum_{i<j}^{n}(x_i - x_j)(y_i - y_j) \tag{17}$$

Before going any further it is useful to make the problem concrete with a low order case, say $n = 3$. For this type of problem we will not be led astray by abnormalities or discontinuities for large n. So for $n = 3$ we have:

$$\begin{aligned}
\sum_{i<j}^{3}(x_i - x_j)(y_i - y_j) =& (x_1 - x_2)(y_1 - y_2) + (x_1 - x_3)(y_1 - y_3) + (x_2 - x_3)(y_2 - y_3)\\
=& x_1y_1 - x_1y_2 - x_2y_1 + x_2y_2 + x_1y_1 - x_1y_3 - x_3y_1 + x_3y_3 + x_2y_2 - x_2y_3 - x_3y_2 + x_3y_3\\
=& 2(x_1y_1 + x_2y_2 + x_3y_3) - (x_1y_2 + x_2y_1 + x_1y_3 + x_3y_1 + x_2y_3 + x_3y_2)
\end{aligned} \tag{18}$$

This suggests that $\sum_{i<j}^{n}(x_i - x_j)(y_i - y_j)$ is the sum of $(n-1)\sum_{i=1}^{n} x_i y_i$ and a negative sum of distinct product terms. In $\sum_{i<j}^{n}(x_i - x_j)(y_i - y_j)$ there are $\frac{n(n-1)}{2}$ pairs where $i < j$ and for each such pair you get $x_i y_i + x_j y_j$ ie 2 such terms. In total you get $\frac{n(n-1)}{2} \times 2 = n(n-1)$ such terms ie $(n-1)$ sums of the form $\sum_{i=1}^{n} x_i y_i$(there being n terms in this sum) ie $(n-1)\sum_{i=1}^{n} x_i y_i$.

The first term on the LHS of (17) is:

$$\frac{1}{2}\sum_{i=1}^{n} 2n x_i y_i = \sum_{i=1}^{n} n x_i y_i \tag{19}$$

(19) overstates the sum of identical terms on the RHS of (17) (ie $(n-1)\sum_{i=1}^{n} x_i y_i$) but we have to adjust for this term:

$$-\frac{1}{2}\Big(x_i \sum_{j=1}^{n} y_j + y_i \sum_{j=1}^{n} x_j\Big) \tag{20}$$

When $i = j$ we have an adjustment:

$$-\frac{1}{2}(x_i y_i + y_i x_i) = -x_i y_i \tag{21}$$

Hence this reduces the sum on the LHS of (17) to (noting that the outer sum from 1 to n applies to the adjustment in (21)):

$$\sum_{i=1}^{n} n x_i y_i - \sum_{i=1}^{n} x_i y_i = (n-1)\sum_{i=1}^{n} x_i y_i \tag{22}$$

which is what we get for product terms of this type in $\sum_{i<j}^{n}(x_i - x_j)(y_i - y_j)$.

This leaves:

$$-\frac{1}{2}\sum_{i=1}^{n}\Big(x_i \sum_{j\neq i}^{n} y_j + y_i \sum_{j\neq i}^{n} x_j\Big) \tag{23}$$

Now (23) reflects double counting of terms $x_i x_j$ where $i < j$. Consider a general term $x_l y_m$ with $l < m$. Such a term gets picked up twice in (23). For, in the outer sum with $i = l$ the first inner sum gives a term $x_l y_m$ and when $i = m$ in the outer sum the second inner sum gives $y_m x_l$, thus giving rise to double counting. Thus (23) collapses to:

$$-\sum_{i=1}^{n}\Big(x_i \sum_{j\neq i}^{n} y_j\Big) \tag{24}$$

which captures all the products in $\sum_{i<j}^{n}(x_i - x_j)(y_i - y_j)$ except those of the form $x_i y_i$ which are caught by (22). To confirm that it works, let's test it for $n = 3$. We get:

$$\begin{aligned} -\sum_{i=1}^{3}\Big(x_i \sum_{j\neq i}^{3} y_j\Big) &= -\Big[x_1(y_2 + y_3) + x_2(y_1 + y_3) + x_3(y_1 + y_2)\Big] \\ &= -\Big(x_1 y_2 + x_1 y_3 + x_2 y_1 + x_2 y_3 + x_3 y_1 + x_3 y_2\Big) \end{aligned} \tag{25}$$

And from (18) the products with $i < j$ in the expansion are:

$$-(x_1 y_2 + x_2 y_1 + x_1 y_3 + x_3 y_1 + x_2 y_3 + x_3 y_2) \tag{26}$$

which is the same as (25).

The terms $x_i y_i$ are picked up by (22) ie

$$(3-1)\sum_{i=1}^{3} x_i y_i = 2\sum_{i=1}^{3} x_i y_i \tag{27}$$

as in (18).

On this basis the last line of (16) does equal this:

$$\frac{1}{n}\sum_{i=1}^{n} x_i y_i = \left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)\left(\frac{1}{n}\sum_{i=1}^{n} y_i\right) + \frac{1}{n^2}\sum_{i<j}(x_i - x_j)(y_i - y_j) \tag{28}$$

There are undoutedly other approaches but this identity is not a straightforward "turn the handle" exercise.

5 Using the identity (9) to prove the sum inequality

It is easy to prove the sum inequality in (1) with (9). Multiplying both sides of (9) by n^2 and noting that since $x_i \le x_j$ and $y_i \le y_j$ for $i < j$ we have:

$$\begin{aligned} n\sum_{i=1}^{n} x_i y_i &= \Big(\sum_{i=1}^{n} x_i\Big)\Big(\sum_{i=1}^{n} y_i\Big) + \sum_{i<j}(x_i - x_j)(y_i - y_j) \\ &\ge \Big(\sum_{i=1}^{n} x_i\Big)\Big(\sum_{i=1}^{n} y_i\Big) \end{aligned} \tag{29}$$

Since $(x_i - x_j)(y_i - y_j) \ge 0$ for all $i < j$.

6 References

[1] Androyer, M., ed. (1883). Cours de M. Hermite, professe pendant le 2e semestre 1881–82, Second tirage revue par M. Hermite, Paris. `https://archive.org/details/coursdemhermite01andogoog`

[2] Adam Besenyei, "Picard's weighty proof of Chebyshev's sum inequality", Mathematics Magazine, 91:5, 366-371, `https://www.tandfonline.com/doi/full/10.1080/0025570X.2018.1512814`

[3] Chebyshev, P. L. (1882). Sur les expressions approximatives des integrales definies par les autres prises entre les meme limites. In: Markov, A. A., Sonin N., eds. (1899–1907) Oeuvres de P. L. Tchebychef, I–II, Vol. 2. St. Petersbourg: Imprimerie de l'Academie imperiale des sciences, pp. 716–719. `https://archive.org/details/117744684_002`

[4] Mark Levi, " The Mathematical Mechanic: Using Physical Reasoning to Solve Problems" Princeton University Press, 2009.

[5] https://www.amazon.com/Mathematical-Mechanic-Physical-Reasoning-Problems/dp/0691154562#customerReviews

7 History

Created

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07 February 2019: corrected typo in (19): y_i instead of y_y and corrected (22) by inserting $-\sum_{i=1}^{n} x_i y_i$ instead of $x_i y_i$ to correctly capture effect of outer sum.