Basic trigonometrical manipulations for analysis

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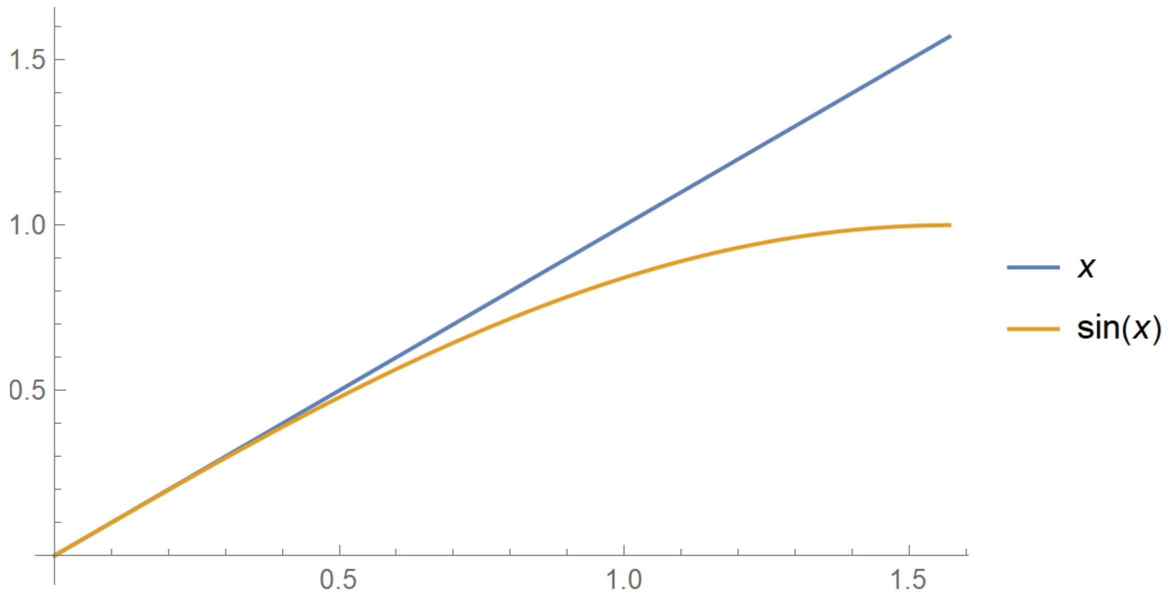
1 Introduction

Analysis, especially Fourier analysis, requires many manipulations of trigonometrical quantities, the most fundamental of which is e^{ix} or its constituent parts $\sin x$ and $\cos x$. This article may be of some use to serious students who want to get their hands a little bit dirtier. Those who really want to plumb the depths can go no further than Antoni Zygmund's influential tome [3]. Zygmund was the PhD supervisor of the late, great Eli Stein who was in turn supervisor of Terry Tao. Eli was a giant in the world of harmonic analysis and a truly nice man to boot. A real loss to the world. We start with some relationships which would be familiar from calculus.

2 The linear bounds of $\sin x$

As they say, a picture speaks a thousand words:



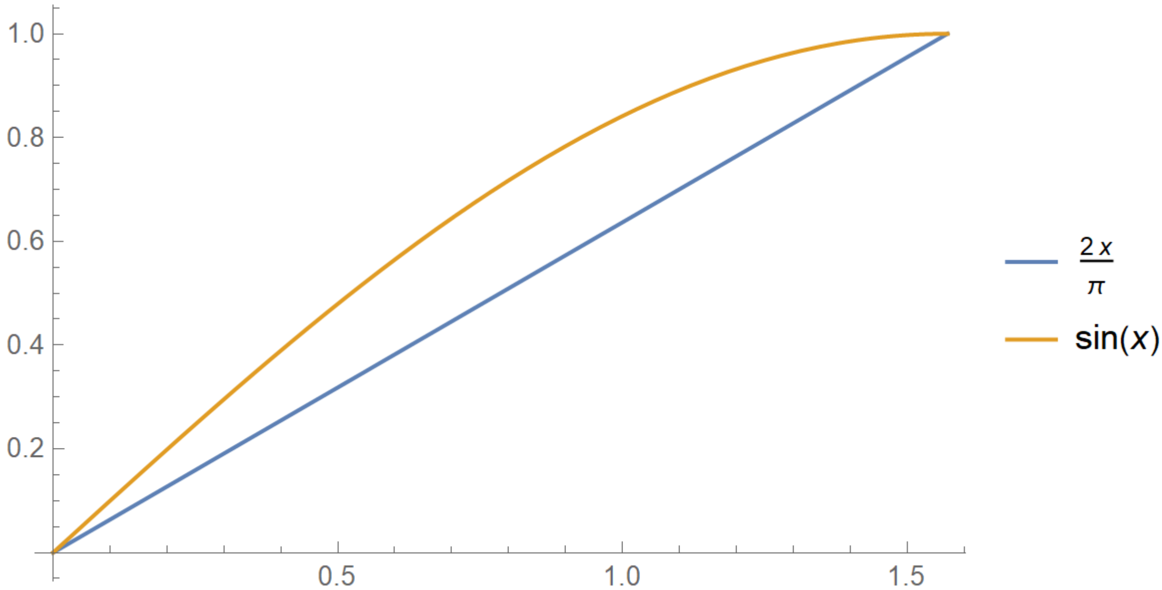
Thus we can see that for $0 \leq x \leq \frac{\pi}{2}$ (actually all $x \geq 0$) that:

$$\boxed{\sin x \leq x} \tag{1}$$

Now this picture is not a proof of the relationship but it is visually compelling. If you needed more convincing you could note that the slope of the line is 1 and since the slope of $\sin x$ is $\cos x$ which decreases from 1 at $x = 0$ to 0 at $x = \frac{\pi}{2}$ the graph of $\sin x$ must lie below the straight line.

Another useful relationship turns upon this graph for $0 \leq x \leq \frac{\pi}{2}$:

$$\boxed{\sin x \geq \frac{2}{\pi}x} \tag{2}$$



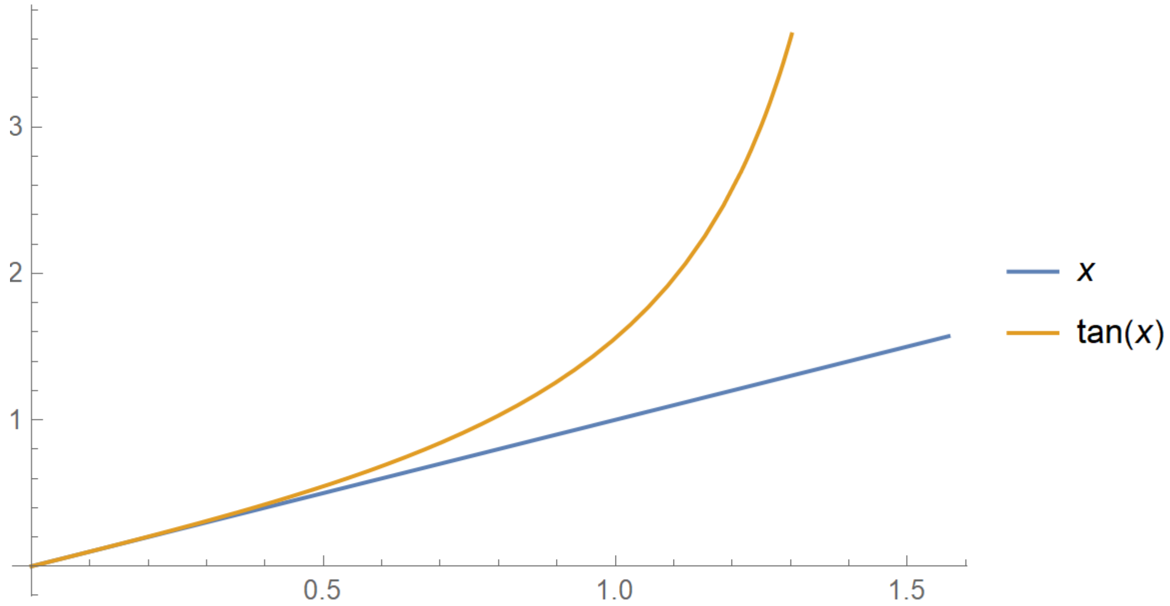
The equation of the straight line from (0,0) to $(\frac{\pi}{2}, 1)$ is simply $y = \frac{2}{\pi}x$.

Now $\sin x$ is concave on $[0, \frac{\pi}{2}]$ as can be seen by looking at its first derivative $\cos x$ which decreases monotonically on $[0, \frac{\pi}{2}]$. Moreover, the second derivative $-\sin x$ is negative on $(0, \frac{\pi}{2}]$ reinforcing that fact.

Another useful relationship for $0 \leq x \leq \frac{\pi}{2}$ is this:

$$\boxed{\tan x \geq \; x} \tag{3}$$

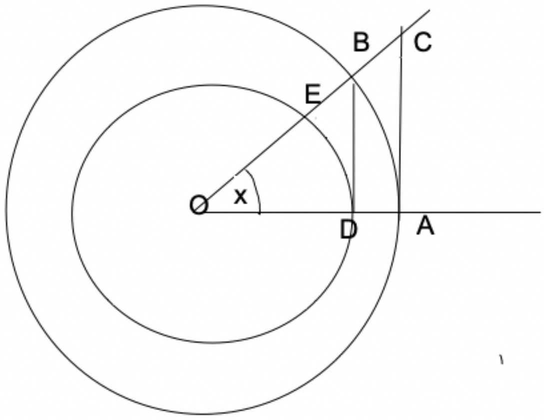
The diagram shows that it is believable:



To prove this it might be thought that one could use the fact that $\frac{1}{\cos x} \geq 1$ for $0 \leq x \leq \frac{\pi}{2}$ but we would then have $\tan x = \frac{\sin x}{\cos x} \geq \sin x \geq \frac{2}{\pi}x$. That relationship may be useful in many contexts but if you know that the Taylor expansion for $\tan x$ is $x + \frac{x^3}{3} + \frac{2x^5}{15} + \dots$ you can see that for $0 \leq x \leq \frac{\pi}{2}$, $\tan x \geq x$. Note that the Taylor series converges in that interval.

Because $\cos x = \sin(\frac{\pi}{2} - x)$ we can get relationships for $\cos x$ as well.

But by far the easiest way to establish the above relationships is via the concept of area which avoids power series and calculus in any overt way. We start with the following diagram:



Here $OB = OA = 1$. DB and AC are perpendiculars. $BD = \sin x$, $OD = \cos x$ and $AC = \tan x$. All we have to do now is compare the areas of the triangles OAB and OAC with the sector OAB. The area of the sector OAB is $\frac{1}{2}1^2 x = \frac{x}{2}$. We thus have the following relationship between the areas:

$$\begin{aligned} \text{Area of } \Delta \text{ OAB } &\leq \frac{x}{2} \leq \text{ Area } \Delta \text{ OAC} \\ \Longrightarrow \frac{1}{2}OA.BD &\leq \frac{x}{2} \leq \frac{1}{2}OA.AC \\ \frac{1}{2}\sin x &\leq \frac{x}{2} \leq \frac{1}{2}\tan x \\ &\boxed{\sin x \leq x \leq \tan x} \end{aligned} \tag{4}$$

This relationship holds for $0 \leq x \leq \frac{\pi}{2}$.

Equation (4) also gives us the important relationship for $x > 0$:

$$\frac{\sin x}{x} \leq 1 \tag{5}$$

The diagram of areas above also enables us to derive the following relationship (see [2], page 72):

$$\boxed{\cos x \leq \frac{\sin x}{x} \leq 1} \tag{6}$$

which is valid for $0 \leq x \leq \frac{\pi}{2}$.

In that diagram the interior circle has radius $OD = \cos x$ so that the sector ODE has area equal to $\frac{1}{2}x \cos^2 x$. But the area of that sector is less than or equal to the area of triangle ODB which has area $\frac{1}{2}OD.BD = \frac{1}{2}\cos x \sin x$. Thus we have:

$$\begin{aligned} \frac{1}{2}x \cos^2 x \leq & \frac{1}{2}\cos x \sin x \\ x\cos x \leq & \sin x \\ \cos x \leq & \frac{\sin x}{x} \end{aligned} \tag{7}$$

The above analysis has been based on the unstated assumption that x is real, but what happens if we are seeking bounds on $\sin z$ where z is complex? The short answer is that $\sin z$ is unbounded for z complex. This can be appreciated from the definition of $\sin z$ for z complex:

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \tag{8}$$

Thus to take a simple example:

$$\begin{aligned} \sin i = & \frac{e^{-1} - e^{1}}{2i} \\ = & \frac{(e^{1} - e^{-1})i}{2} \\ = & i \sinh 1 \end{aligned} \tag{9}$$

But note that:

$$|\sin z| = \left| \frac{(e^{1} - e^{-1})i}{2} \right| = \frac{e^{1} - e^{-1}}{2} > 1 \tag{10}$$

Hardy's book "A Course of Pure Mathametics" ([1] , page 469) contains some problems which give rise to some interesting bounds for the special case where constraints are placed on z. The first bound of interest is that for $|z| < 1$:

$$|\cos z| < 2 \tag{11}$$

We let $z = a + bi$ with $|z| = \sqrt{a^2 + b^2} < 1$ and $b \neq 0$. Thus $|a|, |b| < 1$.

$$
\begin{aligned}
|\cos z| =& \sqrt{\cos z \ \cos \bar{z}} \\
=& \sqrt{\cos(a + ib) \ \cos(a - ib)} \\
=& \sqrt{[\cos a \ \cos(ib) - \sin a \sin(ib)][\cos a \ \cos(ib) + \sin a \sin(ib)]} \\
=& \sqrt{[\cos a \ \cosh b – i \sin a \ \sinh b][\cos a \ \cosh b + i \sin a \ \sinh b]} \\
=& \sqrt{\cos^2 a \ \cosh^2 b + \sin^2 a \ \sinh^2 b} \\
=& \sqrt{\cos^2 a \ \cosh^2 b + (1 - \cos^2 a) \ \sinh^2 b} \\
=& \sqrt{\cos^2 a \underbrace{(\cosh^2 a - \sinh^2 a)}_{=1} + \sinh^2 b} \\
=& \sqrt{\cos^2 a + \sinh^2 b} \\
=& \sqrt{\cos^2 a + \frac{1}{4}(e^{2b} - 2 + e^{-2b})} \\
<& \sqrt{1 + \frac{1}{4}(9 - 2 + 1)} \\
=& \sqrt{3} \\
<& 2
\end{aligned}
\tag{12}
$$

Note here that we have used the facts that $\cos ix = \cosh x$ and $\sin ix = i \sinh x$. These results follow straight from the definitions as follows:

$$
\begin{aligned}
\cos ix =& \frac{e^{-x} + e^{x}}{2} \\
=& \cosh x
\end{aligned}
\tag{13}
$$

Similarly:

$$
\begin{aligned}
\sin ix =& \frac{e^{-x} - e^{x}}{2i} \\
=& i \frac{e^{x} - e^{-x}}{2} \\
=& i \sinh x
\end{aligned}
\tag{14}
$$

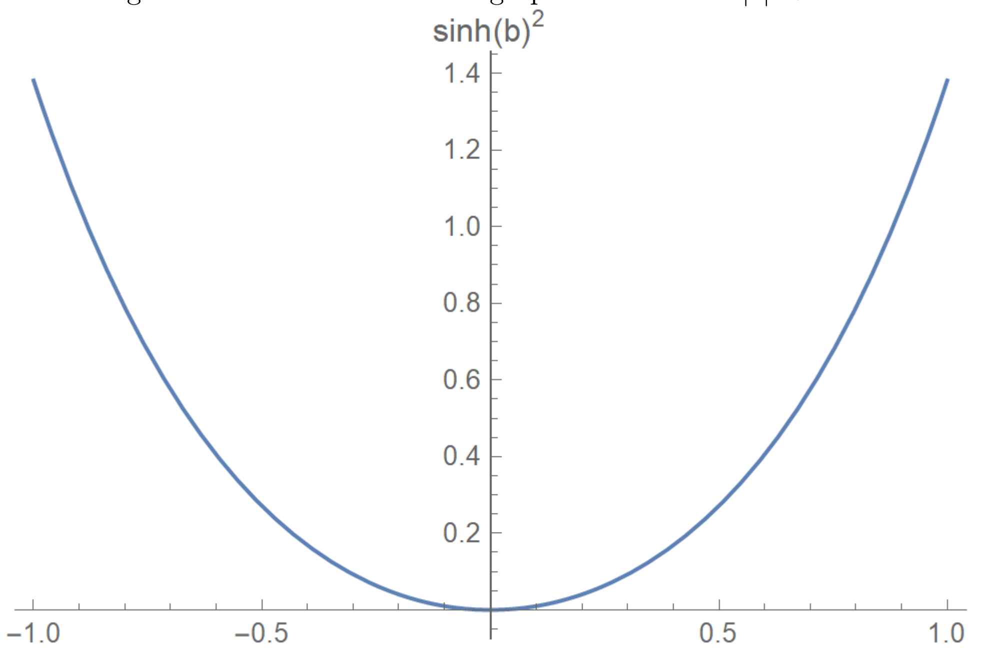
Hardy's next inequality is trickier. For $|z| < 1$ you have to show that:

$$|\sin z| < \frac{6}{5}|z| \tag{15}$$

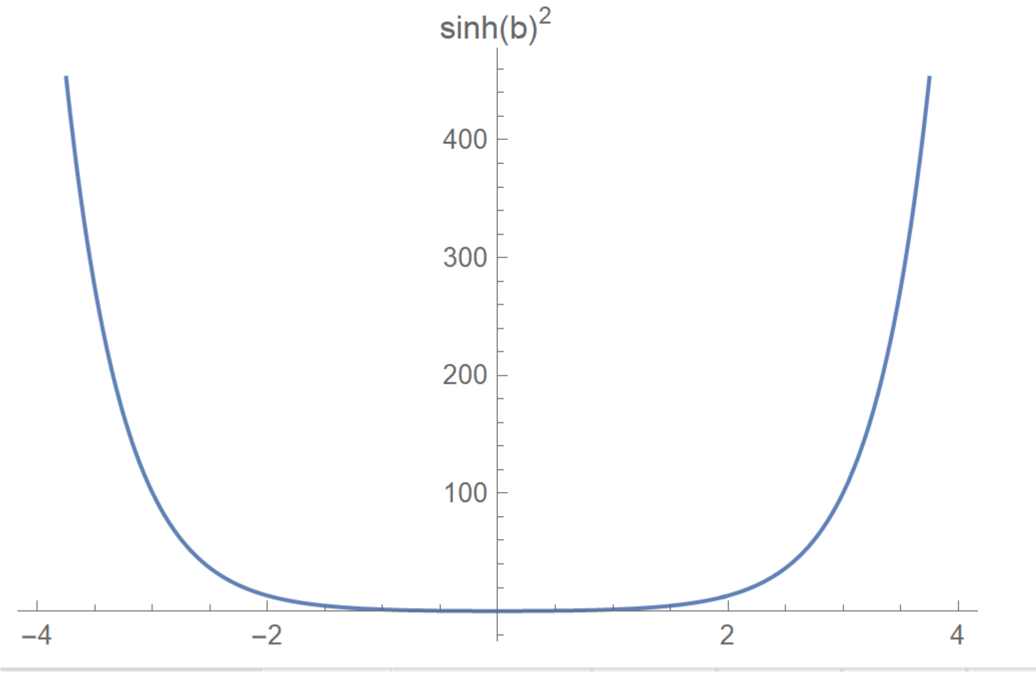
We proceed as before:

$$\begin{aligned}
|\sin z| =& \sqrt{\sin z \, \sin \bar{z}} \\
=& \sqrt{\sin(a+ib) \, \sin(a-ib)} \\
=& \sqrt{[\sin a \, \cos(ib) + \cos a \sin(ib)][\sin a \, \cos(ib) - \cos a \sin(ib)]} \\
=& \sqrt{[\sin a \, \cosh b + i \sin a \, \sinh b][\sin a \, \cosh b - i \cos a \, \sinh b]} \\
=& \sqrt{\sin^2 a \, \cosh^2 b + \cos^2 a \, \sinh^2 b} \\
=& \sqrt{\sin^2 a \, \cosh^2 b + (1 - \sin^2 a) \, \sinh^2 b} \\
=& \sqrt{\sin^2 a + \sinh^2 b} \\
\leq& \sqrt{a^2 + \sinh^2 b}
\end{aligned} \tag{16}$$

At this stage it is useful to look at the graph of $\sinh^2 b$ for $|b| < 1$:



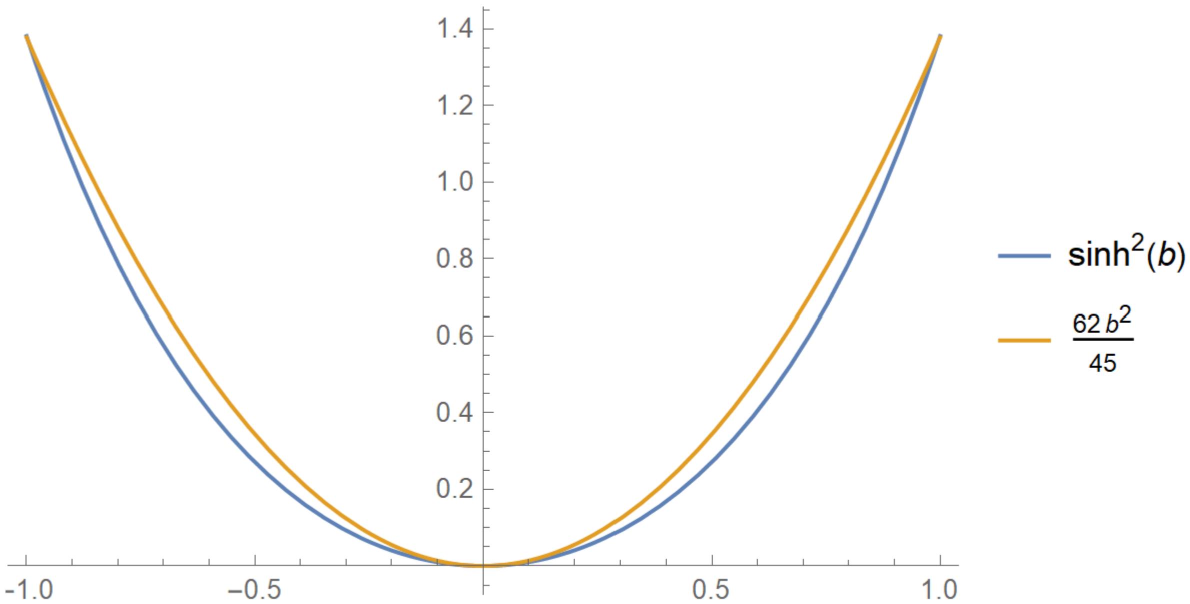
If it looks suspiciously quadratic, that's because for this range of b it is close to quadratic. If we consider $|b| < 4$ the graph look likes this:



We can see the quadratic character by getting the Taylor series of $\sinh^2 b = \frac{e^{2b}-2+e^{-2b}}{4}$ about $b = 0$. The Taylor series is:

$$
\begin{aligned}
\sinh^2 b =& \frac{1}{4}(e^{2b} - 2 + e^{-2b}) \\
=& \frac{1}{4}\Big(2 + 2\frac{(2b)^2}{2!} + 2\frac{(2b)^4}{4!} + 2\frac{(2b)^6}{6!} + \cdots - 2\Big) \\
=& b^2 + \frac{1}{3}b^4 + \frac{2}{45}b^6 + \mathcal{O}(b^7) \\
<& b^2(1 + \frac{1}{3} + \frac{2}{45} + \text{remainder }) \text{ because } |b| < 1 \\
=& \frac{62}{45}b^2 + \text{ remainder}
\end{aligned}
\tag{17}
$$

The graph below shows how close the Taylor series is to the actual function. It looks like it dominates but we need to do some remainder analysis to be sure.



Recall that if we stop the Taylor expansion of $f(x)$ at term k the remainder is bounded by:

$$\left| \frac{f^{(k+1)}(\xi)}{(k+1)!}(x-a)^{k+1} \right| \tag{18}$$

where $a < \xi < x$.

Bearing in mind that in the Taylor expansion of $f(b) = \sinh^2 b$ we are performing the expansion around $b = 0$ for the interval $(-1, 1)$ and we have stopped when $k = 6$, the error term is bounded by $k = 7$ where:

$$f^{(k)}(b) = \begin{cases} 2^{k-2}(e^{2b} - e^{-2b}) \text{ if k is odd} \\ 2^{k-1}(e^{2b} + e^{-2b}) \text{ if k is even} \end{cases} \tag{19}$$

Thus we are looking at a bound of the following form, where we take $\xi = 1$ and $x = 1$

to maximise the bound:

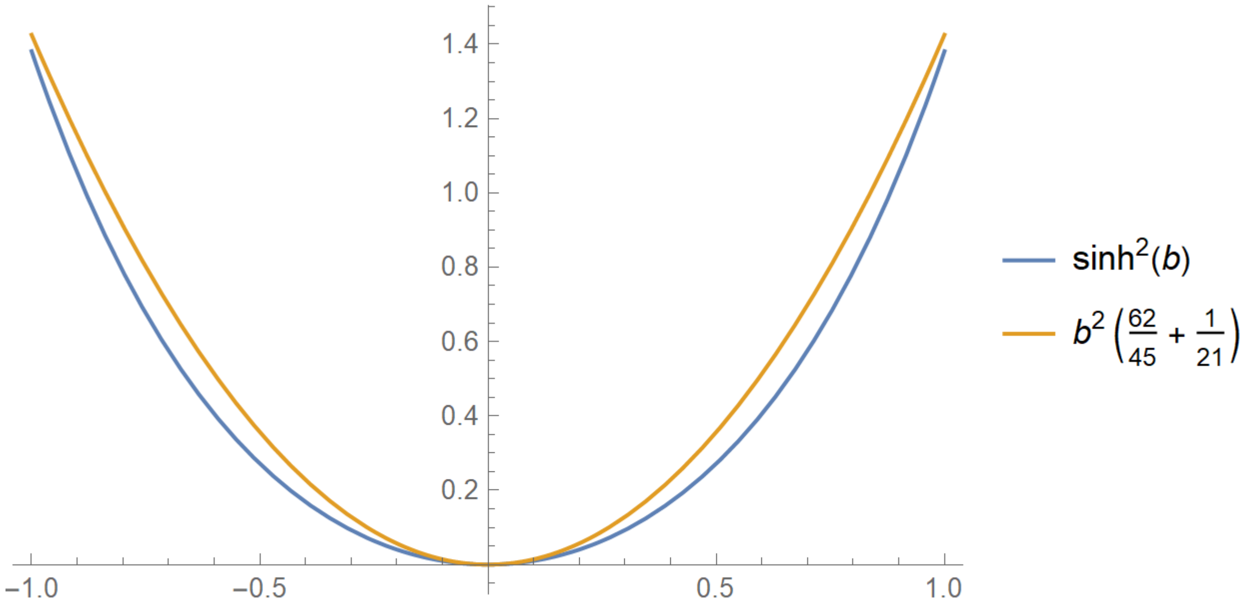
$$\begin{aligned}\left|\frac{f^{(7)}(\xi)}{7!}x^7\right| &\leq \left|\frac{f^{(7)}(\xi)}{7!}\right| \\ &\leq \left|\frac{2^5(e^2 - e^{-2})}{7!}\right| \\ &< \frac{2^5}{7!}\frac{15}{2} \\ &= \frac{1}{21}\end{aligned} \tag{20}$$

So in (17) our remainder which was $\mathcal{O}(x^7)$ actually becomes of $\mathcal{O}(x^2)$ because $b^7 < b^2$ given that $|b| < 1$.

Going back to (17) we have that:

$$\sinh^2 b < b^2(1 + \frac{1}{3} + \frac{2}{45} + \frac{1}{21}) = (\frac{62}{45} + \frac{1}{21})b^2 \tag{21}$$

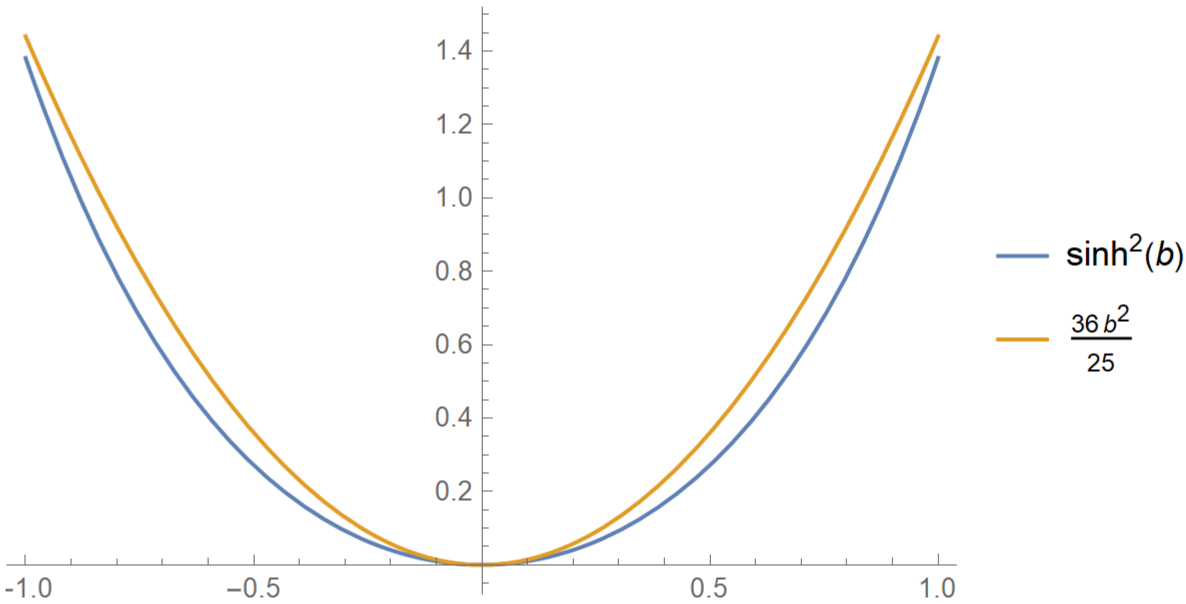
This is what the estimate looks like:



But we can dominate the RHS of (21) by noting that:

$$\frac{62}{45} + \frac{1}{21} < \frac{62}{45} + \frac{1}{20} = \frac{1285}{900} < \frac{1296}{900} = \frac{36^2}{36 \times 25} = \frac{36}{25} \tag{22}$$

The final estimate looks like this:



So going back to (16) we have that:

$$
\begin{aligned}
|\sin z| &< \sqrt{a^2 + \sinh^2 b} \\
&< \sqrt{\frac{36}{25}(a^2 + b^2)} \\
&= \frac{6}{5}|z|
\end{aligned}
\tag{23}
$$

which is what Hardy got.

3 Some hyperbolic manipulations

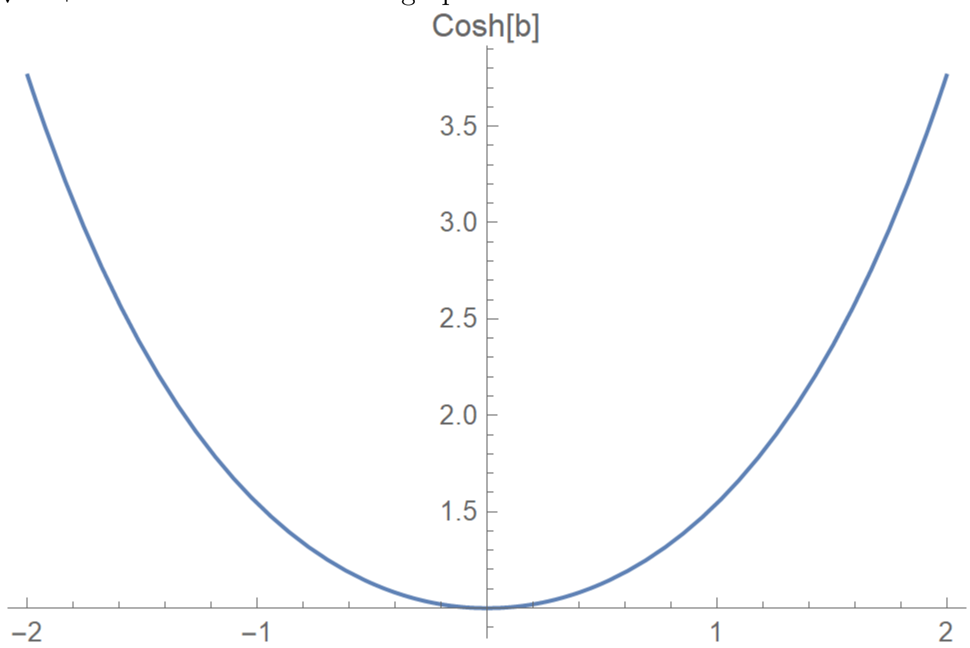
There are two further relationships from Hardy's book ([1], page 469) which are worth checking out:

$$
\begin{aligned}
|\cos z| &\le \cosh|z| \\
|\sin z| &\le \sinh|z|
\end{aligned}
\tag{24}
$$

To satisfy ourselves that these relationships are kosher we proceed as follows. We let $z = a + ib$:

$$\begin{aligned}
|\cos z| &= \sqrt{\cos(a+ib)\ \cos(a-ib)} \\
&= \sqrt{\left[\cos a \cos(ib) - \sin a \sin(ib)\right]\left[\cos a \cos(ib) + \sin a \sin(ib)\right]} \\
&= \sqrt{(\cos a \cosh b - i \sin a \sinh b)\,(\cos a \cosh b + i \sin a \sinh b)} \\
&= \sqrt{\cos^2 a \cosh^2 b + \sin^2 a \sinh^2 b} \\
&= \sqrt{(1-\sin^2 a) \cosh^2 b + \sin^2 a \sinh^2 b} \\
&= \sqrt{\cosh^2 b + \sin^2 a(\sinh^2 b - \cosh^2 b)} \\
&= \sqrt{\cosh^2 b - \sin^2 a} \\
&\le \sqrt{\cosh^2 b} \\
&= \cosh b \\
&\le \cosh \sqrt{a^2+b^2} \\
&= \cosh|z|
\end{aligned} \tag{25}$$

The step which involves the claim that $\cosh b \le \cosh\sqrt{a^2+b^2}$ is justified because $b \le \sqrt{a^2+b^2}$ and the nature of the graph of $\cosh b$:

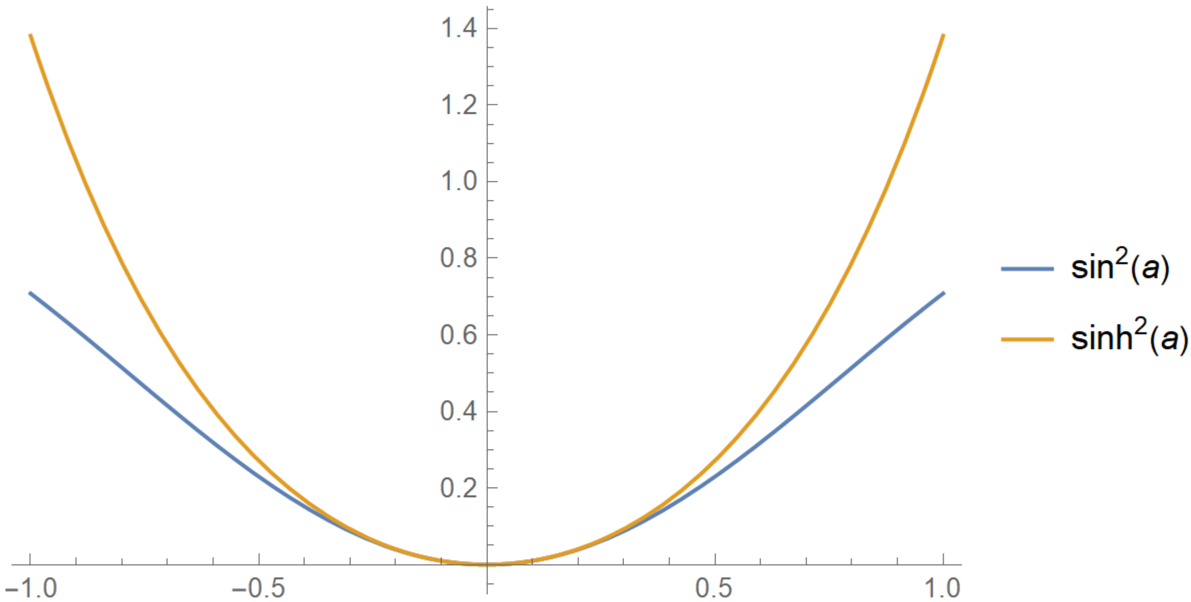


It might be expected that we can apply the same sort of approach to prove that $|\sin z| \le \sinh|z|$ but this quickly proves illusory. Let's just try it:

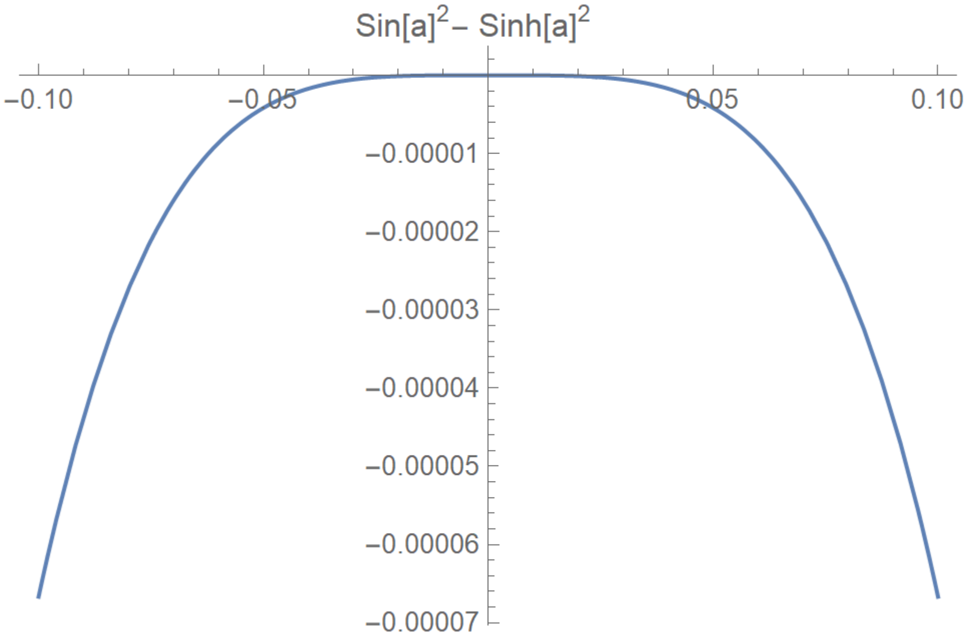
$$\begin{aligned}
|\sin z| &= \sqrt{\sin(a+ib)\,\sin(a-ib)} \\
&= \sqrt{\big[\sin a\cos(ib) + \cos a\sin(ib)\big]\,\big[\sin a\cos(ib) - \cos a\sin(ib)\big]} \\
&= \sqrt{(\sin a\cosh b + i\cos a\sinh b)\,(\sin a\cosh b - i\cos a\sinh b)} \\
&= \sqrt{\sin^2 a\cosh^2 b + \cos^2 a\sinh^2 b} \\
&= \sqrt{\sin^2 a\,(1+\sinh^2 b) + (1-\sin^2 a)\,\sinh^2 b} \\
&= \sqrt{\sin^2 a + \sinh^2 b}
\end{aligned} \tag{26}$$

Now, morally, it should be the case that $\sin^2 a \le \sinh^2 a$ since $\sin^2 a \le a^2$ for all a (not just $0 \le a \le \frac{\pi}{2}$) and $\sinh^2 a$ involves exponentials which will obviously dominate a^2. If there were ever to be any issues it would be when a is very small since both $\sin^2 a$ and $\sinh^2 a$ behave like a^2.

Here is the high level behaviour:



Here is the behaviour for small a:



Analytically we can satisfy ourselves that $\sin^2 a \leq \sinh^2 a$ as follows:

$$\begin{aligned}
\sinh^2 a =& \frac{e^{2a} + e^{-2a} - 2}{4} \\
=& \frac{1}{4}\left(1 + 2a + \frac{(2a)^2}{2!} + \frac{(2a)^3}{3!} + \frac{(2a)^4}{4!} + \dots + 1 - 2a + \frac{(2a)^2}{2!} - \frac{(2a)^3}{3!} + \frac{(2a)^4}{4!} + \dots - 2\right) \\
=& \frac{1}{4}\left(2\frac{(2a)^2}{2!} + 2\frac{(2a)^4}{4!} + 2\frac{(2a)^6}{6!} + \dots\right) \\
=& a^2 + \frac{1}{3}a^4 + \frac{2}{45}a^6 + \dots \\
\geq& a^2 \\
\geq& \sin^2 a
\end{aligned} \tag{27}$$

So at this stage from (26) and (27) we have:

$$|\sin z| \leq \sqrt{\sinh^2 a + \sinh^2 b} \tag{28}$$

Now when we expand $\sinh^2 a$ (and $\sinh^2 b$) by Taylor's series we get:

$$
\begin{aligned}
\sinh^2 a =& \frac{1}{4}(e^{2a} + e^{-2a} - 2) \\
=& \frac{1}{4}\big(1 + 2a + \frac{(2a)^2}{2!} + \frac{(2a)^3}{3!} + \frac{(2a)^4}{4!} + \dots - 2 + 1 - 2a + \frac{(2a)^2}{2!} - \frac{(2a)^3}{3!} + \frac{(2a)^4}{4!} + \dots - 2\big) \\
=& \frac{2}{4}\big(\frac{(2a)^2}{2!} + \frac{(2a)^4}{4!} + \frac{(2a)^6}{6!} + \dots\big)
\end{aligned} \tag{29}
$$

Similarly for $\sinh^2 b$ so that we have:

$$
\begin{aligned}
\sinh^2 a + \sinh^2 b =& \frac{2}{4}\left(\frac{(2a)^2}{2!} + \frac{(2a)^4}{4!} + \frac{(2a)^6}{6!} + \dots + \frac{(2b)^2}{2!} + \frac{(2b)^4}{4!} + \frac{(2b)^6}{6!} + \dots\right) \\
=& \frac{2}{4}\left(\frac{(2a)^2 + (2b)^2}{2!} + \frac{(2a)^4 + (2b)^4}{4!} + \frac{(2a)^6 + (2b)^6}{6!} + \dots\right)
\end{aligned} \tag{30}
$$

The key to getting to our endpoint which is $\sinh \sqrt{a^2 + b^2}$ is to note that:

$$
a^{2k} + b^{2k} \leq \big(\sqrt{a^2 + b^2}\big)^{2k} \tag{31}
$$

and hence:

$$
(2a)^{2k} + (2b)^{2k} \leq \big(2\sqrt{a^2 + b^2}\big)^{2k} \tag{32}
$$

This is because:

$$
(\sqrt{a^2 + b^2})^{2k} = (a^2+b^2)^k = a^{2k}+b^{2k}+ \text{ a sum of non-negative terms of the form } a^2\, b^2 \ \geq a^{2k}+b^{2k} \tag{33}
$$

For instance:

$$
(\sqrt{a^2 + b^2})^4 = (a^2 + b^2)^2 = a^4 + b^4 + 2a^2b^2 \geq a^4 + b^4 \tag{34}
$$

If we do the Taylor expansion of $\sinh^2 \sqrt{a^2+b^2}$ we get:

$$\begin{aligned}\sinh^2 \sqrt{a^2+b^2} =& \frac{1}{4}\left(e^{2\sqrt{a^2+b^2}} + e^{-2\sqrt{a^2+b^2}} - 2\right) \\ =& \frac{2}{4}\left(\frac{(2\sqrt{a^2+b^2})^2}{2!} + \frac{(2\sqrt{a^2+b^2})^4}{4!} + \frac{(2\sqrt{a^2+b^2})^6}{6!}\dots\right)\end{aligned} \tag{35}$$

Recalling (28) and comparing (30) and (35) having regard to (32) we come to this:

$$|\sin z| \leq \sqrt{\sinh^2 + \sinh^2 b} \leq \sqrt{\sinh^2 \sqrt{a^2+b^2}} \leq \sinh|z| \tag{36}$$

3.1 Some more bounds on $|\sin z|$ and $|\cos z|$ for complex z

With $z = x + iy$ the following two inequalities hold:

$$\begin{aligned}\sinh y \leq |\sin z| \leq \cosh y \\ \sinh y \leq |\cos z| \leq \cosh y\end{aligned} \tag{37}$$

From (16) we see that:

$$\begin{aligned}|\sin z| =& \sqrt{\sin^2 x \, \cosh^2 y + \cos^2 x \, \sinh^2 y} \\ =& \sqrt{\sin^2 x \, \cosh^2 y + \cos^2 x \, (\cosh^2 y - 1)} \\ =& \sqrt{(\sin^2 x + \cos^2 x) \cosh^2 y - \cos^2 x} \\ \leq& \sqrt{(\sin^2 x + \cos^2 x) \cosh^2 y} \\ =& \cosh y\end{aligned} \tag{38}$$

Also,

$$\begin{aligned}|\sin z| =& \sqrt{\sin^2 x \, \cosh^2 y + \cos^2 x \, \sinh^2 y} \\ =& \sqrt{\sin^2 x \, (1 + \sinh^2 y) + \cos^2 x \, \sinh^2 y} \\ =& \sqrt{(\sin^2 x + \cos^2 x) \sinh^2 y + \sin^2 x} \\ \geq& \sinh y\end{aligned} \tag{39}$$

Thus the first inequality in (37) is proved. Similarly, from (12) we have that:

$$\begin{aligned} |\cos z| &= \sqrt{\cos^2 x\, \cosh^2 y + \sin^2 x\, \sinh^2 y} \\ &= \sqrt{\cos^2 x\, \cosh^2 y + \sin^2 x\,(\cosh^2 y - 1)} \\ &= \sqrt{\cosh^2 y - \sin^2 x} \\ &\leq \sqrt{\cosh^2 y} \\ &= \cosh y \end{aligned} \tag{40}$$

Finally:

$$\begin{aligned} |\cos z| &= \sqrt{\cos^2 x\, \cosh^2 y + \sin^2 x\, \sinh^2 y} \\ &= \sqrt{\cos^2 x\,(1 + \sinh^2 y) + \sin^2 x\, \sinh^2 y} \\ &= \sqrt{\sinh^2 y + \cos^2 x} \\ &\geq \sqrt{\sinh^2 y} \\ &= \sinh y \end{aligned} \tag{41}$$

thus proving the second inequality of (37).

4 Manipulating e^{ix}

In the context of Fourier theory the Fourier transform takes on the following form (there are others which are equivalent but this is the one I will be using):

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i \xi x}\, dx \tag{42}$$

If we are asked to prove the uniform continuity of the Fourier transform we are inevitably led to obtaining an estimate for something like this:

$$|e^{ia} - e^{ib}| \tag{43}$$

Recall that uniform continuity is a *global* rather than a local property in the sense that uniform continuity refers to a *set* of points rather than just one point. The logical statement of the property is as follows and it is useful to understand the order of the quantifiers because they are important:

$$\forall \epsilon > 0, \forall \xi, \eta \in \mathrm{X}, \ \exists \delta > 0 \text{ such that } |f(\xi) - f(\eta)| < \epsilon \text{ whenever } |\xi - \eta| < \delta \tag{44}$$

The main concept to take away is that with uniform continuity, you can give me any positive epsilon and I can find a delta for any two points in the set such that the absolute value of the difference in the Fourier transforms evaluated at those two points is less than epsilon.

In order to prove the theorem we have to get to a stage of showing that $|\hat{f}(\xi) - \hat{f}(\eta)| < \epsilon$ when $|\xi - \eta| < \delta$ which means that we have to somehow arrive at something that looks like this:

$$|\hat{f}(\xi) - \hat{f}(\eta)| < \text{constant } \times |\xi - \eta| \tag{45}$$

because if we make $|\xi - \eta| < \delta$ where $\delta \leq \frac{1}{\text{constant}} \epsilon$ then (45) guarantees that $|\hat{f}(\xi) - \hat{f}(\eta)| < \epsilon$.

Thus we need (43) in a form which helps with (45) and indeed we can show that for any real a, b:

$$\boxed{|e^{ia} - e^{ib}| \leq |a - b|} \tag{46}$$

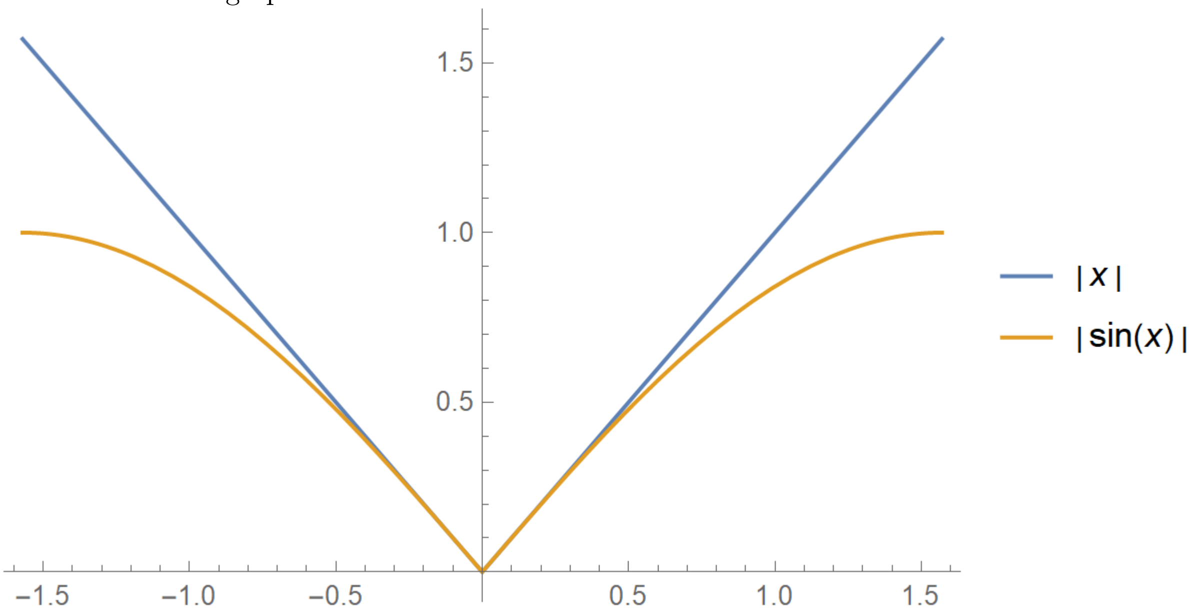
This is proved as follows:

$$\begin{aligned}
|e^{ia} - e^{ib}| =& |\cos a + i \sin a - \cos b - i \sin b| \\
=& |\cos a - \cos b + i(\sin a - \sin b)| \\
=& \sqrt{(\cos a - \cos b)^2 + (\sin a - \sin b)^2} \\
=& \sqrt{\cos^2 a - 2 \cos a \ \cos b + \cos^2 b + \sin^2 a - 2 \sin a \ \sin b + \sin^2 b} \\
=& \sqrt{2(1 - \cos(a - b))} \\
=& \sqrt{2 \times 2 \sin^2 \left(\frac{a-b}{2} \right)} \\
=& \leq 2 \times \left| \sin \left(\frac{a-b}{2} \right) \right| \\
\leq& 2 \times \frac{|a-b|}{2} \\
=& |a - b|
\end{aligned} \tag{47}$$

Note that with $b = 0$ in (46) we have:

$$|e^{ia} - 1| \leq |a| \tag{48}$$

Note that (46) also encapsulates the fact that $|\sin a| \leq |a|$ for all a. Indeed it would be a bit of a worry if it didn't and you can convince yourself by performing the calculation or savour the graph below:



4.1 A compact result used in Fejer's theorem

In his textbook Körner proves ([4], page 7) the positivity of the Fejèr kernel by a method that is very slick. He starts the derivation with the following proposition:

$$\sum_{r=-n}^{n} (n+1-|r|)e^{irx} = \Big(\sum_{k=0}^{n} e^{i(k-\frac{n}{2})x}\Big)^2 \tag{49}$$

That this is so is not immediately obvious and, typically, he gives no hint. It is undoubtedly one of the many little mathematical 'engines' that Cambridge Tripos students (Körner 's audience) would be familiar with. It may even be due to Fejer himself but I haven't gone back and looked at his papers. Let us consider the following polynomial:

$$p_n(x) = \Big(\sum_{k=0}^{n} x^k\Big)^2 \tag{50}$$

We can play with some low values of n to get the following format which looks like Pascal's Triangle:

$$\begin{aligned}(x^0)^2 &= \boxed{1} \\ (1+x)^2 &= \boxed{1} + \boxed{2}x + \boxed{1}x^2 \\ (1+x+x^2)^2 &= \boxed{1} + \boxed{2}x + \boxed{3}x^2 + \boxed{2}x^3 + \boxed{1}x^4 \\ (1+x+x^2+x^3)^2 &= \boxed{1} + \boxed{2}x + \boxed{3}x^2 + \boxed{4}x^3 + \boxed{3}x^4 + \boxed{2}x^5 + \boxed{1}x^6\end{aligned} \tag{51}$$

Thus we guess that for $n \geq 1$:

$$p_n(x) = \sum_{k=0}^{n}(k+1)x^k + \sum_{k=n+1}^{2n}(2n-k+1)x^k \tag{52}$$

This is proved inductively as follows. The proposition is true for $n = 1$:

$p_1(x) = 1+2x+x^2$ while $\sum_{k=0}^{1}(k+1)x^k + \sum_{k=2}^{2}(2-k+1)x^k = 1+2x+x^2$. Assuming the proposition is true for any n we have:

$$
\begin{aligned}
p_{n+1}(x) =& \Big(\sum_{k=0}^{n+1} x^k\Big)^2 \\
=& \Big(\sum_{k=0}^{n} x^k + x^{n+1}\Big)^2 \\
=& \Big(\sum_{k=0}^{n} x^k\Big)^2 + 2x^{n+1}\sum_{k=0}^{n} x^k + x^{2n+2} \\
=& \underbrace{\sum_{k=0}^{n}(k+1)x^k + \sum_{k=n+1}^{2n}(2n-k+1)x^k}_{\text{using the induction hypothesis}} + 2\sum_{k=0}^{n} x^{k+n+1} + x^{2n+2} \\
=& \sum_{k=0}^{n}(k+1)x^k + \sum_{k=n+2}^{2n+2}(2n-k+1)x^k + nx^{n+1} + x^{2n+2} + 2x^{n+1} + 2\sum_{k=1}^{n} x^{k+n+1} + x^{2n+2} \\
=& \sum_{k=0}^{n+1}(k+1)x^k + \sum_{k=n+2}^{2n+2}(2n-k+1)x^k + 2\sum_{k=1}^{n} x^{k+n+1} + 2x^{2n+2} \\
=& \sum_{k=0}^{n+1}(k+1)x^k + \sum_{k=n+2}^{2n+2}(2n-k+1)x^k + 2\underbrace{\sum_{k=n+2}^{2n+1} x^k}_{k+n+1\to k} + 2x^{2n+2} \\
=& \sum_{k=0}^{n+1}(k+1)x^k + \sum_{k=n+2}^{2n+2}(2n-k+1)x^k + 2\sum_{k=n+2}^{2n+2} x^k \\
=& \sum_{k=0}^{n+1}(k+1)x^k + \sum_{k=n+2}^{2n+2}[2(n+1)-k+1]x^k
\end{aligned}
\tag{53}
$$

Hence the proposition is true for $n+1$. We can now apply this general result to $\Big(\sum_{k=0}^{n} e^{i(k-\frac{n}{2})x}\Big)^2$:

$$
\begin{aligned}
\Big(\sum_{k=0}^{n} e^{i(k-\frac{n}{2})x}\Big)^2 =& e^{-inx}\Big(\sum_{k=0}^{n} e^{ikx}\Big)^2 \\
=& e^{-inx}\Big(\sum_{k=0}^{n}(k+1)e^{ikx} + \sum_{k=n+1}^{2n}[2n-k+1]e^{ikx}\Big) \\
=& \underbrace{\sum_{k=0}^{n}(k+1)e^{i(k-n)x}}_{\text{low}} + \underbrace{\sum_{k=n+1}^{2n}[2n-k+1]e^{i(k-n)x}}_{\text{high}}
\end{aligned}
\tag{54}
$$

The coefficient of e^{irx} for $1 \leq r \leq n$ comes from the "high" representation in (24) so that $k - n = r$. Hence the coefficient is $2n - k + 1 = 2n - (r + n) + 1 = n + 1 - r$. Similarly the coefficient of e^{irx} for $-n \leq r \leq -1$ comes from the "low representation" so that $k - n = r$ and so $k + 1 = n + r + 1 = n + 1 - |r|$. When $r = 0$, $k = n$ and the coefficient of e^0 is $n + 1 = n + 1 - r$. So in all cases the coefficient of e^{irx} is of the form $n + 1 - |r|$.

Hence $\sum_{r=-n}^{n}(n+1-|r|)e^{irx} = \Big(\sum_{k=0}^{n} e^{i(k-\frac{n}{2})x}\Big)^2$. We can now derive the desired result as follows:

$$
\begin{aligned}
\sum_{r=-n}^{n} \frac{(n+1-|r|)}{n+1} e^{irx} =& \frac{1}{n+1}\Big(\sum_{k=0}^{n} e^{i(k-\frac{n}{2})x}\Big)^2 \\
=& \frac{1}{n+1}\Big(e^{-\frac{inx}{2}} \sum_{k=0}^{n} e^{ikx}\Big)^2 \\
=& \frac{1}{n+1}\Big(e^{-\frac{inx}{2}} \frac{(1-e^{i(n+1)x})}{1-e^{ix}}\Big)^2 \\
=& \frac{1}{n+1}\Big(e^{-\frac{inx}{2}} \frac{(e^{-\frac{i(n+1)x}{2}} e^{\frac{i(n+1)x}{2}} - e^{\frac{i(n+1)x}{2}} e^{\frac{i(n+1)x}{2}})}{(e^{\frac{ix}{2}} e^{-\frac{ix}{2}} - e^{\frac{ix}{2}} e^{\frac{ix}{2}})}\Big)^2 \\
=& \frac{1}{n+1}\left(e^{-\frac{inx}{2}} e^{\frac{i(n+1)x}{2}} \frac{(e^{-\frac{i(n+1)x}{2}} - e^{\frac{i(n+1)x}{2}})}{e^{\frac{ix}{2}}(e^{-\frac{ix}{2}} - e^{\frac{ix}{2}})}\right)^2 \\
=& \frac{1}{n+1}\left(\frac{e^{-\frac{i(n+1)x}{2}} - e^{\frac{i(n+1)x}{2}}}{e^{-\frac{ix}{2}} - e^{\frac{ix}{2}}}\right)^2 \\
=& \frac{1}{n+1}\left(\frac{-2i\sin(\frac{(n+1)x}{2})}{-2i\sin(\frac{x}{2})}\right)^2 \\
=& \frac{1}{n+1}\left(\frac{\sin(\frac{(n+1)x}{2})}{\sin(\frac{x}{2})}\right)^2
\end{aligned}
\tag{55}
$$

5 Euler's product expansion for the sine

Euler is rightly famous for his many outrageous results and his original works still consume researchers. For many years US mathematician Ed Sandifer has been running the Euler archive for Mathematical Association of America members (`http://eulerarchive.maa.org`) but there is also a new source of Euler edification, namely, Euleriana: `https://scholarlycommons.pacific.edu/euleriana/`

Euler's infinite product for the sine is:

$$\frac{\sin x}{x} = \prod_{j=1}^{\infty} \left(1 - \frac{x^2}{j^2\pi^2}\right) \tag{56}$$

There are various proofs of this result but one relies upon the following theorem: For all complex numbers z and all integers n we have:

$$\boxed{\sin nz = 2^{n-1} \sin z \sin\left(z + \frac{\pi}{n}\right) \sin\left(z + \frac{2\pi}{n}\right) \ldots \sin\left(z + \frac{(n-1)\pi}{n}\right)} \tag{57}$$

To prove (57) involves a common exponential manipulation. We write $\sin nz$ as follows:

$$\begin{aligned} \sin nz =& \frac{e^{inz} - e^{-inz}}{2i} \\ =& \frac{e^{2inz} - 1}{2ie^{inz}} \\ =& e^{-inz} \frac{e^{2inz} - 1}{2i} \end{aligned} \tag{58}$$

Now we can expand $e^{2inz} - 1$ by noting that it has the form $u^n - 1$ with $u = e^{2iz}$.

The roots of $u^n = 1$ are:

$$u_k = e^{-\frac{2k\pi i}{n}} \tag{59}$$

for $k = 0, 1, 2, \ldots (n-1)$.

Thus we can expand $u^n - 1$ as a product of its roots as follows:

$$e^{2inz} - 1 = (e^{2iz} - 1)\,(e^{2iz} - e^{-\frac{2\pi i}{n}})\,(e^{2iz} - e^{-\frac{4\pi i}{n}})\,\ldots(e^{2iz} - e^{-\frac{2(n-1)\pi i}{n}}) \tag{60}$$

There are n factors in this product and going back to (58) we have:

$$\sin nz = e^{-inz} \frac{(e^{2iz} - e^{-\frac{2\pi i}{n}})\,(e^{2iz} - e^{-\frac{4\pi i}{n}})\,\ldots(e^{2iz} - e^{-\frac{2(n-1)\pi i}{n}})}{2i} \tag{61}$$

Next we manipulate each of the n factors on the RHS of (61) as follows:

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} = \frac{e^{2iz} - 1}{2ie^{iz}} \implies e^{2iz} - 1 = 2ie^{iz} \sin z \tag{62}$$

$$\sin\Big(z+\frac{\pi}{n}\Big) = \frac{e^{i(z+\frac{\pi}{n})} - e^{-i(z+\frac{\pi}{n})}}{2i} = \frac{e^{2i(z+\frac{\pi}{n})} - 1}{2ie^{i(z+\frac{\pi}{n})}} \implies e^{2iz} - e^{-\frac{2\pi i}{n}} = 2ie^{i(z+\frac{\pi}{n})}\, \frac{\sin(z+\frac{\pi}{n})}{e^{\frac{2\pi i}{n}}} \tag{63}$$

and so on for the k^{th} factor (note that $k = 0, 1, 2, \ldots, (n-1)$:

$$\sin\Big(z+\frac{k\pi}{n}\Big) = \frac{e^{i(z+\frac{k\pi}{n})} - e^{-i(z+\frac{k\pi}{n})}}{2i} = \frac{e^{2i(z+\frac{k\pi}{n})} - 1}{2ie^{i(z+\frac{k\pi}{n})}} \implies e^{2iz} - e^{-\frac{2k\pi i}{n}} = 2ie^{i(z+\frac{k\pi}{n})}\, \frac{\sin(z+\frac{k\pi}{n})}{e^{\frac{2k\pi i}{n}}} \tag{64}$$

All we have to do now is plug in each of the n factors from (64) into (61):

$$\begin{aligned}
\sin nz =& \frac{e^{-inz}}{2i} \prod_{k=0}^{n-1} \Big(2ie^{iz}\, e^{-\frac{k\pi i}{n}} \sin(z+\frac{k\pi}{n})\Big) \\
=& \frac{e^{-inz}}{2i}\, (2i)^n e^{inz}\, e^{-\frac{\pi i(1+2+\cdots+(n-1))}{n}} \prod_{k=0}^{n-1} \sin(z+\frac{k\pi}{n})) \\
=& (2i)^{n-1} e^{-\frac{(n-1)\pi i}{2}} \prod_{k=0}^{n-1} \sin(z+\frac{k\pi}{n})) \\
=& 2^{n-1} e^{\frac{(n-1)\pi i}{2}}\, e^{-\frac{(n-1)\pi i}{2}} \prod_{k=0}^{n-1} \sin(z+\frac{k\pi}{n})) \\
=& 2^{n-1} \sin z\, \sin(z+\frac{\pi}{n})\, \sin(z+\frac{2\pi}{n}) \ldots \sin(z+\frac{(n-1)\pi}{n})
\end{aligned} \tag{65}$$

Note that $i = e^{\frac{\pi i}{2}}$.

In (57) if we divide by $\sin z$ we get this:

$$\frac{nz}{\sin z}\, \frac{\sin nz}{nz} = 2^{n-1} \sin(z+\frac{\pi}{n})\, \sin(z+\frac{2\pi}{n}) \ldots \sin(z+\frac{(n-1)\pi}{n}) \tag{66}$$

Now if we let $z \to 0$ we get:

$$\sin(z+\frac{\pi}{n})\, \sin(z+\frac{2\pi}{n}) \ldots \sin(z+\frac{(n-1)\pi}{n}) = \frac{n}{2^{n-1}} \tag{67}$$

since $\frac{\sin nz}{nz} \to 1$ $\frac{z}{\sin z} \to 1$ and as $z \to 0$.

6 Trigonometric series and power series

The fact that trigonometric series are real parts of power series often suggests a way of summing them. Consider a sum like this:

$$S = \frac{1}{2} + z + z^2 + z^3 + \dots \tag{68}$$

where $z = re^{ix}$ and $0 \leq r < 1$.

We see that:

$$(S - \frac{1}{2})(1 - z) = z \implies S = \frac{1}{2}\frac{1+z}{1-z} \tag{69}$$

We can simply the RHS of (69) as follows:

$$\begin{aligned} \frac{1}{2}\frac{1+re^{ix}}{1-re^{ix}} &= \frac{1}{2}\left(\frac{1+re^{ix}}{1-re^{ix}}\right)\left(\frac{1-re^{-ix}}{1-re^{-ix}}\right) \\ &= \frac{1}{2}\left(\frac{1+r(e^{ix}-e^{-ix})-r^2}{1-r(e^{ix}+e^{-ix})+r^2}\right) \\ &= \frac{1}{2}\left(\frac{1+2ri\sin x - r^2}{1-2r\cos x + r^2}\right) \end{aligned} \tag{70}$$

Thus we have that the real part of the RHS of (70) is:

$$\mathcal{R}e\left(\frac{1}{2}\frac{1+z}{1-z}\right) = \frac{1}{2}\left(\frac{1-r^2}{1-2r\cos x + r^2}\right) \tag{71}$$

and the imaginary part is:

$$\mathcal{I}m\left(\frac{1}{2}\frac{1+z}{1-z}\right) = \frac{r\sin x}{1-2r\cos x + r^2} \tag{72}$$

Zygmund provides a trickier example of the same technique ([3], page 2). He starts with this formula for $z = re^{ix}$ where $0 \leq r < 1$:

$$\log\frac{1}{1-z} = z + \frac{1}{2}z^2 + \dots + \frac{1}{n}z^n + \dots \tag{73}$$

Note that log here is the logarithmic function applying to a complex number rather than ln which is usually reserved for the natural logarithm of a positive real number.

To prove (73) you start with the Taylor expansion of $f(z) = \log(1-z)$ about $z = 0$. The derivatives are found to be:

$$f^{(k)}(z) = -\frac{(k-1)!}{(1-z)^k} \tag{74}$$

Hence $f^{(k)}(0) = -(k-1)!$ for $k = 1, 2, 3, \dots$ and $f(0) = 0$. Thus:

$$\begin{aligned}\log(1-z) =& f(0) + zf^{(1)}(0) + \frac{z^2}{2!}f^{(2)}(0) + \dots + \frac{z^k}{k!}f^{(k)}(0) + \dots \\ =& -z - \frac{1}{2}z^2 - \frac{1}{3}z^3 - \dots - \frac{1}{k}z^k - \dots \\ =& -\sum_{k=1}^{\infty}\frac{1}{k}z^k\end{aligned} \tag{75}$$

So using (75), $\log\frac{1}{1-z} = \log(1-z)^{-1} = -\log(1-z) = \sum_{k=1}^{\infty}\frac{1}{k}z^k$. Zygmund states that you then get:

$$\sum_{k=1}^{\infty}\frac{\cos kx}{k}r^k = \frac{1}{2}\log\frac{1}{1-2r\cos x + r^2} \tag{76}$$

and

$$\sum_{k=1}^{\infty}\frac{\sin kx}{k}r^k = \arctan\frac{r\sin x}{1-r\cos x} \tag{77}$$

with $\arctan 0 = 0$.

At this stage is worthwhile reviewing some complex variable theory (see [5], pages 39-40). If z is a complex number with is not equal to zero, $\log z$ is defined to be a number which has the following property that $e^w = z$ and thus we say that:

$$w = \log z \tag{78}$$

Now since $z = |z|e^{i(\theta+2n\pi)}$ where $\theta = \arg z$ and $n \in \mathbb{Z}$ we then have:

$$w = \log|z| + i(\theta + 2n\pi) \tag{79}$$

This is a multivalued function and we can define the principal value of $\log z$ to be $\log|z| + i\Theta$ where $-\pi < \Theta \leq \pi$.

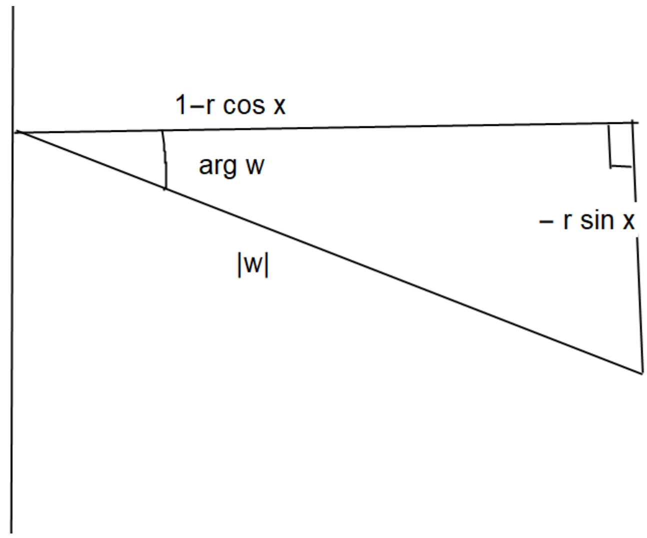
In the context of this problem we have, noting $z = re^{ix}$:

$$\begin{aligned} \log \frac{1}{1-z} &= -\log(1-z) \\ &= -\log(1 - r\cos x - ir\sin x) \\ &= -\log w \end{aligned} \tag{80}$$

Now

$$\begin{aligned} |w| &= |1 - r\cos x - ir\sin x| \\ &= \sqrt{(1 - r\cos x - ir\sin x)\,(1 - r\cos x + ir\sin x)} \\ &= \sqrt{(1 - r\cos x)^2 + r^2\sin^2 x} \\ &= \sqrt{1 - 2r\cos x + r^2} \end{aligned} \tag{81}$$

In (79) let us take $n = 0$ and work out $\theta = \arg w$ from the following diagram:



We have:

$$\tan(\arg w) = \frac{-r\sin x}{1 - r\cos x} \implies \arg w = \arctan\left(\frac{-r\sin x}{1 - r\cos x}\right) = -\arctan\left(\frac{r\sin x}{1 - r\cos x}\right) \tag{82}$$

So going back to (80) and (79) (with $n = 0$) we have that:

$$\begin{aligned} \log\frac{1}{1-z} &= -\log\sqrt{1 - 2r\cos x + r^2} - i\theta \\ &= \frac{1}{2}\log\frac{1}{1 - 2r\cos x + r^2} + i\arctan\left(\frac{r\sin x}{1 - r\cos x}\right) \end{aligned} \tag{83}$$

since arctan is odd.

Finally, we note from (73) that:

$$\begin{aligned} \log\frac{1}{1-z} &= \sum_{k=1}^{\infty}\frac{z^k}{k} \\ &= \sum_{k=1}^{\infty}\frac{e^{kix}}{k}r^k \\ &= \sum_{k=1}^{\infty}\frac{\cos kx}{k}r^k + i\sum_{k=1}^{\infty}\frac{\sin kx}{k}r^k \end{aligned} \tag{84}$$

Equating real and imaginary parts of (83) and (84) we finally get (76) and (77). When $r = 0$ or $x = 0$, $\arctan(0) = 0$ and both sides of (76) and (77) make sense.

6.1 A bound on a sum of complex exponentials

In Fourier theory one finds sums such as this:

$$\sum_{k=M+1}^{M+N} e^{2\pi kit} \tag{85}$$

Now we can get an easy bound on this sum as follows:

$$\Big|\sum_{k=M+1}^{M+N} e^{2\pi kit}\Big| \le \sum_{k=M+1}^{M+N} |e^{2\pi kit}| \le \sum_{k=M+1}^{M+N} 1 = N \tag{86}$$

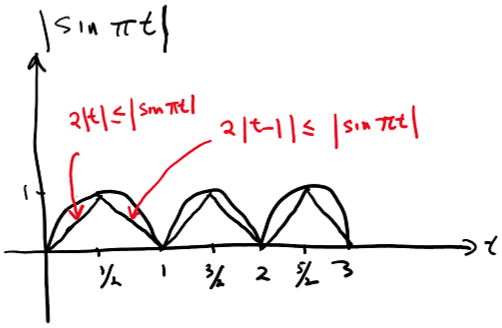
there being $M + N - (M+1) + 1 = N$ summands in the sum.

However, this is such a crude sum because of the intellectually impoverished way it was arrived at. A more subtle bound requires more subtle reasoning. We thus proceed as follows:

$$\begin{aligned}
\Big|\sum_{k=M+1}^{M+N} e^{2\pi kit}\Big| &= \Big|e^{2\pi(M+1)it}\left(1 + e^{2\pi it} + e^{4\pi it} + \cdots + e^{2(N-1)\pi it}\right)\Big| \\
&= \Big|e^{2\pi(M+1)it}\ \frac{(1 - e^{2\pi iNt})}{1 - e^{2\pi it}}\Big| \\
&= \Big|e^{2\pi(M+1)it}\ \frac{e^{N\pi it}(e^{-N\pi it} - e^{N\pi it})}{e^{\pi it}(e^{-\pi it} - e^{\pi it})}\Big| \\
&= \Big|e^{2\pi(M+1)it} e^{(N-1)\pi it}\frac{(e^{-N\pi it} - e^{N\pi it})}{(e^{-\pi it} - e^{\pi it})}\Big| \\
&= \Big|e^{2\pi(M+1)it} e^{(N-1)\pi it}\Big|\ \Big|\frac{(e^{-N\pi it} - e^{N\pi it})}{(e^{-\pi it} - e^{\pi it})}\Big| \\
&= \Big|\frac{e^{-N\pi it} - e^{N\pi it}}{e^{-\pi it} - e^{\pi it}}\Big| \\
&= \Big|\frac{2i\sin(N\pi t)}{2i\sin(\pi t)}\Big| \\
&= \Big|\frac{\sin(N\pi t)}{\sin(\pi t)}\Big| \\
&\le \frac{1}{|\sin(\pi t)|}
\end{aligned} \tag{87}$$

We just note here that $|e^{ix}| = 1$ which justifies $\left|e^{2\pi(M+1)it}e^{(N-1)\pi it}\right| = 1$.

The following graph enables us to refine things a bit more:



If we let $\|t\| = \min\{|t-k| : k \in \mathbb{Z}\}$ which tells us how far t is from the nearest integer, the above graph justifies us saying that:

$$2\|t\| \leq |\sin(\pi t)| \implies \frac{1}{2\|t\|} \geq \frac{1}{|\sin(\pi t)|} \tag{88}$$

Putting (88) together with (86) our more subtle bound is:

$$\Big|\sum_{k=M+1}^{M+N} e^{2\pi kit}\Big| \leq \min\{N, \frac{1}{|\sin(\pi t)|}\} \leq \min\{N, \frac{1}{2\|t\|}\} \tag{89}$$

7 References

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[2] Milton Ambramowitz and Irene A Stegun, editors, Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables National Bureau of Standards, Applied Mathematics Series, 55, Tenth Printing, 1972
[3] Antoni Zygmund, Trigonometric Series , Third Edition, Volumes I and II combined, Cambridge University Press, 2002.

[4] T W Körner, Fourier Analysis, Cambridge University Press, 1990.
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8 History

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