Completing the square

An important application in probability theory

Completing the square in the context of Gaussian distributions

- **Background**

I was tripping the light fantastic through Norbert Wiener's book "**Nonlinear Problems in Random Theory", John Wiley, 1958** and on page 2 he has a derivation which uses the technique of completing the square which is important to know in the context of Gaussian distributions.

He starts with a quantity x that has a Gaussian distribution ie the probability that x is between x_1 and x_2 is:

$$\int_{x_1}^{x_2} \frac{1}{\sqrt{2\pi a}} e^{-\left(\frac{x^2}{2a}\right)} dx$$

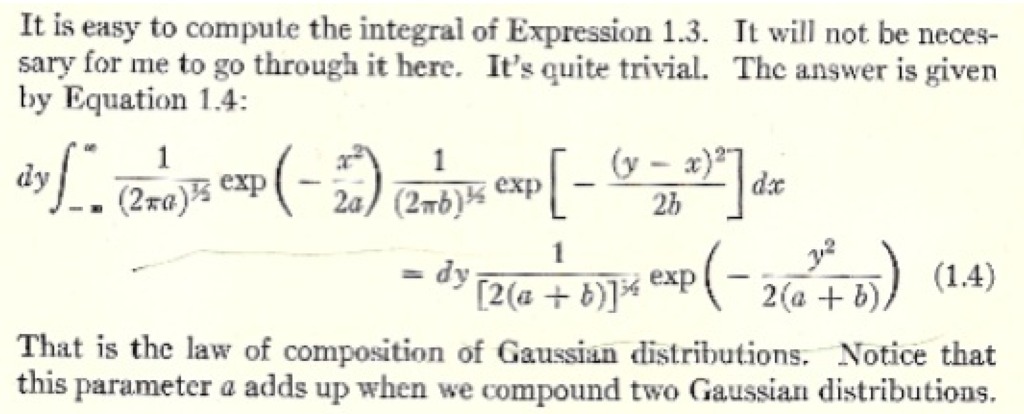
He then composes two distributions via this joint distribution:

$$\frac{1}{\sqrt{2\pi a}} e^{-\left(\frac{x^2}{2a}\right)} \frac{1}{\sqrt{2\pi b}} e^{-\left(\frac{(y-x)^2}{2b}\right)}$$

Next he is interested in this:

$$dy \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi a}} e^{-\left(\frac{x^2}{2a}\right)} \frac{1}{\sqrt{2\pi b}} e^{-\left(\frac{(y-x)^2}{2b}\right)} dx$$ and he says the answer is:

$$dy \frac{1}{\sqrt{2(a+b)}} e^{-\left(\frac{y^2}{2(a+b)}\right)}$$



It is easy to compute the integral of Expression 1.3. It will not be necessary for me to go through it here. It's quite trivial. The answer is given by Equation 1.4:

$$dy \int_{-\infty}^{\infty} \frac{1}{(2\pi a)^{1/2}} \exp\left(-\frac{x^2}{2a}\right) \frac{1}{(2\pi b)^{1/2}} \exp\left[-\frac{(y-x)^2}{2b}\right] dx = dy \frac{1}{[2(a+b)]^{1/2}} \exp\left(-\frac{y^2}{2(a+b)}\right) \quad (1.4)$$

That is the law of composition of Gaussian distributions. Notice that this parameter a adds up when we compound two Gaussian distributions.

This is the law of the composition of Gaussian distributions, but as you will see there is a typo in the formula.

The correct answer is $dy \frac{1}{\sqrt{2\pi(a+b)}} e^{-\left(\frac{y^2}{2(a+b)}\right)}$

Completing the square

In essence we have to perform this integration:

$$I = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi a}}\, e^{-\left(\frac{x^2}{2a}\right)} \frac{1}{\sqrt{2\pi b}}\, e^{-\left(\frac{(y-x)^2}{2b}\right)} dx = \frac{1}{\sqrt{2\pi a}} \; \frac{1}{\sqrt{2\pi b}} \int_{-\infty}^{\infty} e^{-\left(\frac{x^2}{2a}\right) - \left(\frac{(y-x)^2}{2b}\right)} dx$$

Now $-\left(\frac{x^2}{2a}\right) - \frac{(y-x)^2}{2b} = \frac{-bx^2 - a(y^2 - 2xy + x^2)}{2ab}$

$$= \frac{-(a+b)x^2 - ay^2 + 2axy}{2ab}$$

$$= \frac{-(a+b)}{2ab}\left[x^2 - \frac{2ay}{a+b}x + \frac{a}{a+b}y^2 \right]$$

$$= \frac{-(a+b)}{2ab}\left[\left(x - \frac{ay}{a+b}\right)^2 - \left(\frac{ay}{a+b}\right)^2 + \frac{a}{a+b}y^2 \right]$$

$$= \frac{-(a+b)}{2ab}\left\{ \left(x - \frac{ay}{a+b}\right)^2 - \frac{\left[a^2y^2 - (a+b)ay^2\right]}{(a+b)^2} \right\}$$

$$= \frac{-(a+b)}{2ab}\left\{ \left(x - \frac{ay}{a+b}\right)^2 + \frac{aby^2}{(a+b)^2} \right\}$$

$$= \frac{-(a+b)}{2ab}\left(x - \frac{ay}{a+b}\right)^2 - \frac{y^2}{2(a+b)}$$

So the integral becomes:

$$I = \frac{1}{\sqrt{2\pi a}} \; \frac{1}{\sqrt{2\pi b}} \int_{-\infty}^{\infty} e^{\frac{-(a+b)}{2ab}\left(x - \frac{ay}{a+b}\right)^2 - \frac{y^2}{2(a+b)}} dx$$

$$= \frac{1}{\sqrt{2\pi a}} \; \frac{1}{\sqrt{2\pi b}} \; e^{-\frac{y^2}{2(a+b)}} \int_{-\infty}^{\infty} e^{\frac{-(a+b)}{2ab}\left(x - \frac{ay}{a+b}\right)^2} dx$$ since y is a constant

Let $u = \sqrt{\frac{(a+b)}{2ab}}\left(x - \frac{ay}{a+b}\right)$

Then $du = \sqrt{\frac{(a+b)}{2ab}}\, dx$

Hence $\int_{-\infty}^{\infty} e^{\frac{-(a+b)}{2ab}\left(x - \frac{ay}{a+b}\right)^2} dx = \frac{1}{\sqrt{\frac{(a+b)}{2ab}}} \int_{-\infty}^{\infty} e^{-u^2}\, du$

Now $\int_{-\infty}^{\infty} e^{-u^2}\, du = \sqrt{\pi}$

If you don't know how to work this out the standard trick is as follows:

Let $J = \int_{-\infty}^{\infty} e^{-u^2}\, du$

Then $J^2 = \left(\int_{-\infty}^{\infty} e^{-u^2}\, du\right)\left(\int_{-\infty}^{\infty} e^{-v^2}\, dv\right)$

$= \iint_P e^{-(u^2+v^2)}\, du\, dv$ where the integration is over the whole plane P.

Now you switch from rectangular to polar co-ordinates (ie u = r cos θ and v = r sin θ). Thus $u^2 + v^2 = r^2$ and du dv = r dr dθ. More formally the integral $\iint_P e^{-(u^2+v^2)}\, du\, dv = \iint_Q e^{-r^2} \left|\frac{\partial(u,v)}{\partial(r,\theta)}\right|$ dr dθ where

$$\left|\frac{\partial(u,v)}{\partial(r,\theta)}\right| = \left|\begin{pmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial \theta} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial \theta} \end{pmatrix}\right| = \left|\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}\right| = r$$

So $\iint_P e^{-(u^2+v^2)}\, du\, dv = \iint_Q e^{-r^2} \left|\frac{\partial(u,v)}{\partial(r,\theta)}\right|$ dr dθ = $\int_0^{2\pi}\int_0^{\infty} e^{-r^2}$ r dr dθ

$= \int_0^{2\pi} \left[\frac{-1}{2} e^{-w}\right]_0^{\infty} d\theta$ where w = -r^2 so dw = -2r dr

$= \int_0^{2\pi} \frac{1}{2}\, d\theta$

$= \pi$

So $J^2 = \pi$ hence J = $\sqrt{\pi}$ as claimed.

Thus $\frac{1}{\sqrt{\frac{(a+b)}{2ab}}} \int_{-\infty}^{\infty} e^{-u^2}\, du = \frac{\sqrt{\pi}}{\sqrt{\frac{(a+b)}{2ab}}}$

So finally I = $\frac{1}{\sqrt{2\pi a}} \frac{1}{\sqrt{2\pi b}} e^{-\frac{y^2}{2(a+b)}} \int_{-\infty}^{\infty} e^{\frac{-(a+b)}{2ab}\left(x - \frac{ay}{a+b}\right)^2}$

$= \frac{1}{\sqrt{2\pi a}} \frac{1}{\sqrt{2\pi b}} e^{-\frac{y^2}{2(a+b)}} \frac{\sqrt{\pi}}{\sqrt{\frac{(a+b)}{2ab}}}$

$= \frac{\sqrt{\pi}\,\sqrt{2ab}}{\sqrt{2\pi a}\,\sqrt{2\pi b}\,\sqrt{a+b}} e^{-\frac{y^2}{2(a+b)}}$

$= \frac{1}{\sqrt{2\pi(a+b)}} e^{-\frac{y^2}{2(a+b)}}$

You will see the same approach in verifying that the bivariate normal distribution has the required characteristics. Thus let the 2 dimensional random variable (X,Y) have the joint probabiity density function:

$$f_{X,Y}(x,y) = f(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} e^{\frac{-\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - 2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right) + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right]}{2(1-\rho^2)}}$$

where the parameters are the usual “suspects” of normal probability theory.

You have to show that $I = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, dx\, dy = 1$

Start with the substitutions:

u = $\frac{x-\mu_X}{\sigma_X}$ so that du = $\frac{dx}{\sigma_X}$

v = $\frac{y-\mu_Y}{\sigma_Y}$ so that dv = $\frac{dy}{\sigma_Y}$

Hence I = $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} e^{\frac{-[u^2-2\rho uv+v^2]}{2(1-\rho^2)}} \sigma_X\ \sigma_Y\ du\ dv$

= $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi\sqrt{1-\rho^2}} e^{\frac{-[(u-\rho v)^2+(1-\rho^2)v^2]}{2(1-\rho^2)}}\ du\ dv$

Now let w = $\frac{u-\rho v}{\sqrt{1-\rho^2}}$ so that dw = $\frac{du}{\sqrt{1-\rho^2}}$

The double integral can now be written as the product of two integrals (that this can be done is proven rigourously in Analysis courses) as follows:

I = $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\frac{-w^2}{2}}\ dw \times \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\frac{-v^2}{2}}\ dv$

= 1 x 1 = 1

History :
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