Celestial mechanics and circular reasoning

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I was reading a good technical paper by an Australian defence scientist, Don Koks, titled "Changing Coordinates in the Context of Orbital Mechanics" [3] and because my celestial mechanics was rusty I picked up some textbooks and refreshed a few ideas. One of the concepts is the basic result that if the position of a particle over time is described by the vector function $\vec{r}(t)$ then the rate at which the radial vector sweeps out the area is given by:

$$\frac{dA}{dt} = \frac{1}{2}r^2(t)\frac{d\theta}{dt} \tag{1}$$

The result is proved in [1] (an excellent book) as follows. Given a circle with centre at the origin and radius r, the area of the sector swept out by the radius as it turns through an angle $\Delta\theta$ is given by:

$$\Delta A = \frac{1}{2}r^2\,\Delta\theta \tag{2}$$

The author of [1], David Bressoud, gives no proof.

Now this is stated as an equality, not an approximation, and it got me wondering how mathematicians and physicists might approach the proof of this basic result. How would you prove it?

The swept area is clearly proportional to the angular displacement and in functional terms it is also clear that the function is continuous. We know that when the angular displacement is 2π radians the area is πr^2. Hence the swept area with angular displacement of 1 radian must be $\frac{\pi r^2}{2\pi} = \frac{r^2}{2}$. So for an angular displacement of $\Delta\theta$ radians the swept area is indeed equal to $\frac{1}{2}r^2\,\Delta\theta$.

We could equally assume that the element of swept area is:

$$dA = \alpha\,d\theta \tag{3}$$

Where α is an unknown constant to be determined. This equation merely expresses the reasonable proposition that the swept area is a linear function of angular displacement.

Now we simply integrate as follows:

$$
\begin{aligned}
A &= \int_0^{2\pi} \alpha \, d\theta \\
\pi r^2 &= 2\pi \, \alpha \\
\therefore \alpha &= \frac{r^2}{2}
\end{aligned} \tag{4}
$$

Thus we get the same result (2).

Bressoud goes on to show that if r is not constant the result dA/dt still applies and the proof necessarily revolves around continuity. It runs like this. If r is not constant then we can find two points in $[s, t]$, call them t_1 and t_2 where r takes on its minimum and maximum values over the interval. Thus on $[s, t]$, $r(t_1) \leq r \leq r(t_2)$

From this it follows that :

$$
\frac{r(t_1)^2}{2} \, \Delta\theta \leq \Delta A \leq \frac{r(t_2)^2}{2} \, \Delta\theta \tag{5}
$$

Now divide through by $\Delta t = t - s$:

$$
\frac{r(t_1)^2}{2} \, \frac{\Delta\theta}{\Delta t} \leq \frac{\Delta A}{\Delta t} \leq \frac{r(t_2)^2}{2} \, \frac{\Delta\theta}{\Delta t} \tag{6}
$$

So as $s \to t$, both t_1 and t_2 approach t and we get:

$$
\frac{r(t_1)^2}{2} \, \frac{d\theta}{dt} \leq \frac{dA}{dt} \leq \frac{r(t_2)^2}{2} \, \frac{d\theta}{dt} \tag{7}
$$

Now let's suppose we don't see this functional approach and we try to approximate the area of the sector by an inscribed triangle. See Figure 1.

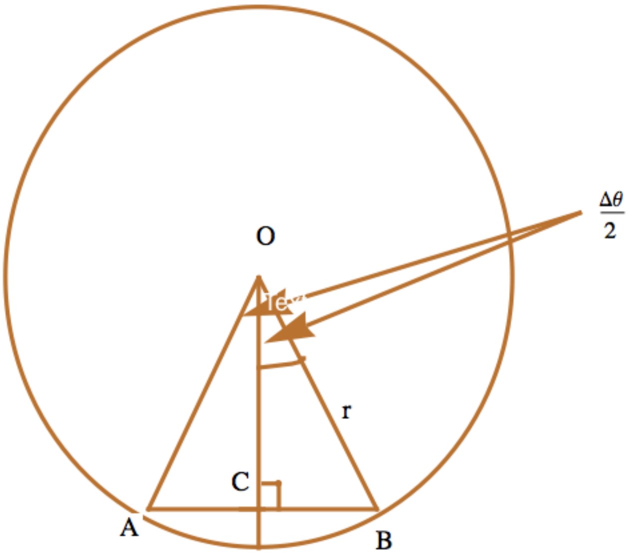

Figure 1

The swept area has total angular displacement $\Delta\theta$ but $\angle BOC = \angle AOC = \frac{\Delta\theta}{2}$. The area of triangle OCB is:

$$\begin{aligned} \text{Area} =& \frac{1}{2} \times r \sin\left(\frac{\Delta\theta}{2}\right) \times r \cos\left(\frac{\Delta\theta}{2}\right) \\ =& \frac{1}{4} r^2 \sin\Delta\theta \end{aligned} \tag{8}$$

Hence the area of triangle OAB is $\frac{1}{2} r^2 \sin\Delta\theta$.

If we now assume that $\Delta\theta$ is small, then $\sin\Delta\theta \approx \Delta\theta$ and so we get that the area of the sector is *approximately* $\frac{1}{2} r^2 \Delta\theta$.

We can replicate this approximate argument by subdividing a circle of radius r into n sectors with central angle $\frac{2\pi}{n}$ so that each inscribed triangle has area $\frac{1}{2} r^2 \sin\frac{2\pi}{n}$. Hence the total area is:

$$n\,\frac{1}{2} r^2 \sin\frac{2\pi}{n} = \pi r^2 \frac{\sin\left(\frac{2\pi}{n}\right)}{\frac{2\pi}{n}} \to \pi r^2 \text{ as } n \to \infty \tag{9}$$

Equation (2) is true for any angular value but the area given by $\frac{1}{2} r^2 \sin\Delta\theta$ is only ever any an approximation.

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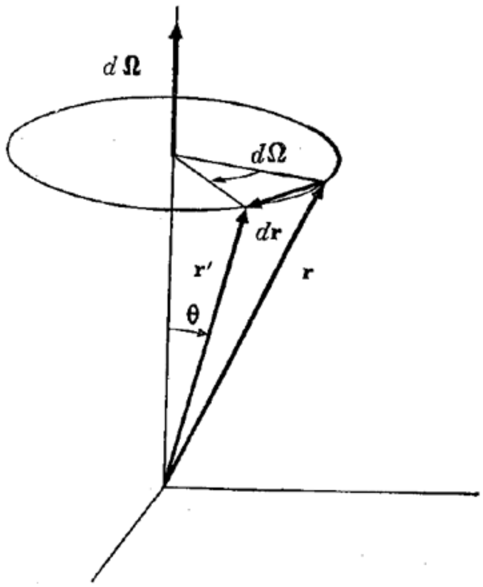

Fig. 4-12. Change in a vector produced by an infinitesimal rotation.

sponding to a counterclockwise rotation of the coordinates). From Fig. 4-12, the magnitude of $d\mathbf{r}$, to first order in $d\Omega$, is

$$dr = r \sin\theta \, d\Omega,$$

which agrees with the magnitude of $\mathbf{r} \times d\mathbf{\Omega}$. Further, $d\mathbf{r}$ must be perpendicular to $d\mathbf{\Omega}$ and $\mathbf{r}$. Finally, the sense is correct, as is shown by Fig. 4-12.

Conversely, the fact that an infinitesimal orthogonal transformation can be written in the form of Eq. (4-94) constitutes an independent proof of Euler's theorem. Any finite displacement of a rigid body with one point fixed can be built up as a succession of infinitesimal displacements. If an infinitesimal transformation is a rotation then the finite transformation must likewise correspond to a rotation.

Figure 2

The approximation is relevant, however, in many other contexts. For instance in considering the change in a vector $\vec{r}$ which is rotated clockwise by a small angle $d\vec{\Omega}$ about the z-axis, the magnitude of $d\vec{r}$ to first order in $d\Omega$ is (see Figure 1 and [2], pages 131-132)

$$dr = r \sin\theta \, d\Omega \tag{10}$$

This is precisely the magnitude of the cross product:

$$\vec{r} \ \mathbf{x} \ d\vec{\Omega} \tag{11}$$

Note that the elemental change in $\vec{r}$ is $d\vec{r}$ and is based on an approximate which uses the cosine rule as follows. By construction, in the sectorial triangle the radii are $r\sin\theta$ thus by the cosine rule we have:

$$\begin{aligned} dr^2 =& 2r^2 \sin^2\theta - 2r^2 \sin^2\theta \, \cos d\Omega \\ =& 2r^2 \sin^2\theta \, (1 - \cos d\Omega) \\ =& 2r^2 \sin^2\theta \, \times \, 2\sin^2\left(\frac{d\Omega}{2}\right) \\ =& 4r^2 \sin^2\theta \, \times \, \sin^2\left(\frac{d\Omega}{2}\right) \\ \therefore dr =& \, 2r\sin\theta \, \sin\left(\frac{d\Omega}{2}\right) \end{aligned} \tag{12}$$

So for $d\Omega$ small, $\sin\left(\frac{d\Omega}{2}\right) \approx \frac{d\Omega}{2}$ and hence the last line of (12) becomes $dr = r\sin\theta \, d\Omega$ as advertised in (10).

1 References

[1] David M Bressoud, *"Second Year Calculus - From Celestial Mechanics to Special Relativity"*, Springer, 1991
[2] Herbert Goldstein, *"Classical Mechanics"*, Wesley, 1950
[3] Don Koks, *"Changing Coordinates in the Context of Orbital Mechanics"* , DST-Group–TN–1594, Defence Science and Technology , January 2017 `https://www.dst.defence.gov.au/sites/default/files/publications/documents/DST-Group-TN-1594.pdf`

2 History

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