Deriving the Stefan-Boltzmann law from Planck's law

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1 Background

The purpose of this short article is to explain the integral that underpins the Stefan-Boltzmann law, being a limiting case of Planck's law of radiation. Undergraduate physics courses almost invariably present Planck's Law, without much discussion, if any, as to how he arrived at it. They leave that to the historians of physics and the definitive explanation of that process is probably given by Max Jammer in [1] (pages 1-21). Jammer was an experimental physicist who also did the history of science in his long life (he died at age 95 in 2010). He was a colleague of Einstein and the material in [1] is based on discussions with, and papers written by, all the "heavy hitters" of 20^{th} century quantum theory.

Jammer explains that in essence what Planck did was to take two expressions of how entropy was related to the energy of an harmonic oscillator at temperarture T and concoct a "compromise". Specifically (and you should read Jammer's book to follow the detail), Planck had these two expressions to work with:

$$\frac{\partial^2 S}{\partial U^2} = \frac{\text{constant}}{U} \tag{1}$$

and

$$\frac{\partial^2 S}{\partial U^2} = \frac{\text{constant}}{U^2} \tag{2}$$

Here S is the entropy and U is the energy. What Planck assumed was that:

$$\frac{\partial^2 S}{\partial U^2} = \frac{a}{U(U+b)} \tag{3}$$

where a, b are constants.

Thus when U is large (3) approximates (2) and when it is small it approximates (1).

Jammer says the following ([1], page 14) about this act of inspiration:

"This interpolation, though mathematically a mere trifle, was one of the most significant and momentous contributions ever made in the history of physics. Not only did it lead Planck, in his search for its logical corroboration, to the proposal of his elementary quantum of action and thus initiated the early development of quantum theory, as we shall see presently; it affected decisively the very foundations of physics as well as their epistemological presuppositions. Never in the history of physics was there such an inconspicuous mathematical interpolation with such far-reaching physical and philosophical consequences."

From (3) Planck deduced that:

$$\frac{1}{T} = \frac{\partial S}{\partial U} = a^{'} \log\left(\frac{U+b}{U}\right) \tag{4}$$

or

$$U = \frac{1}{e^{\left(\frac{1}{a^{'}T}\right)} - 1} \tag{5}$$

where $a^{'} = \frac{-a}{b}$ are functions of frequency v. Using some other results Planck concluded that:

$$u_v = \frac{Av^3}{e^{\left(\frac{Bv}{T}\right)} - 1} \tag{6}$$

where A, B are constants and u_v is energy density at frequency v.

What is typically done in physics courses is that it is observed how Planck's law reduces to the Rayleigh-Jeans formula in the low frequency limit and to Wien's formula in the high frequency limit.

In addition it is usually left as an exercise (look at various forums frequented by physics students if you doubt me) to prove that Stefan-Boltzmann law follows from an appropriate integration of (6).

Recall that the Stefan-Boltzmann law states that the total energy flux Φ emitted from a blackbody at temperature T is:

$$\Phi = \sigma T^4 \tag{7}$$

where σ is a constant.

2 The nittgy gritty of the integration

Since this article is really about how the integration works in detail, I will simply set the problem up as is done in many places such as Wikipedia. Just keep in mind there are various equivalent ways of expressing the constants involved in Planck's law. We start with the intensity of the light emitted from a blackbody in accordance with Planck's law:

$$I(\nu, T) = \frac{2h\nu^3}{c^2} \frac{1}{e^{\left(\frac{h\nu}{kT}\right)} - 1} \tag{8}$$

The power radiated by a surface of area A through a solid angle $d\Omega$ in the differential frequency range $(\nu, \nu + d\nu)$ is:

$$I(\nu, T) A \, d\nu \, d\Omega \tag{9}$$

where:

$$d\Omega = \sin\theta \, d\theta \, d\phi \tag{10}$$

where θ is the angle from the North Pole and ϕ is the longitude.

The Stefan-Boltzmann law says that the power emitted per unit area of the emitting body is:

$$\frac{P}{A} = \int_0^\infty I(\nu, T) \, d\nu \int \cos\theta \, d\Omega \tag{11}$$

Note that the cosine integral appears because black bodies are Lambertian (i.e. they obey Lambert's cosine law), meaning that the intensity observed along the sphere will be the actual intensity times the cosine of the zenith angle (ie angle from the North Pole). Hence we have:

$$\begin{aligned}
\frac{P}{A} = & \int_0^\infty I(\nu, T) \, d\nu \underbrace{\int_0^{2\pi} d\phi \int_0^{\pi/2} \cos\theta \, \sin\theta \, d\theta}_{=2\pi\times\pi/2} \\
=& \pi \int_0^\infty I(\nu, T) \, d\nu \\
=& \pi \int_0^\infty \frac{2h\nu^3}{c^2} \frac{1}{e^{\left(\frac{h\nu}{kT}\right)} - 1} \, d\nu \\
=& \frac{2\pi k^3 T^3}{h^2 c^2} \int_0^\infty \frac{\left(\frac{h\nu}{kT}\right)^3}{e^{\left(\frac{h\nu}{kT}\right)} - 1} \, d\nu
\end{aligned} \tag{12}$$

So what we have to integrate is of the form $\int_0^\infty \frac{u^3}{e^u-1}\,du$. There are various ways of calculating this integral, some of which involve the Riemann Zeta function eg see [2] . Many physics textbooks just say it is a well known result and just state the answer.

Now the first question you have to ask yourself from a purely mathematical nature is whether $\int_0^\infty \frac{u^3}{e^u-1}\,du$ converges. A physicist wouldn't spend too much time on such a question because it is clear that because an exponential grows faster than any power the ratio in the integral decays so rapidly that convergence is assured. That is a perfectly acceptable reason. What about the behaviour at $u = 0$?

In that context the physicist would simply note that:

$$\frac{u^3}{e^u - 1} = \frac{u^3}{1 + u + \frac{u^2}{2!} + \cdots - 1} = \frac{u^2}{1 + \frac{u}{2!} + \ldots} \tag{13}$$

So far from blowing up at $u = 0$, it is simply 0.

Now suppose you are being questioned by an anally retentive mathematician who says "Prove it is convergent!". What is your actual proof?

Looking at (13) we have:

$$1 + \frac{u}{2!} + \frac{u^2}{3!} + \cdots + \frac{u^n}{(n+1)!} + \cdots \geq 1 + \frac{u}{2} + \frac{1}{2!}\left(\frac{u}{2}\right)^2 + \cdots + \frac{1}{n!}\left(\frac{u}{2}\right)^n + \cdots = e^{\frac{u}{2}} \tag{14}$$

since $\frac{u^n}{(n+1)!} \geq \frac{1}{n!}\left(\frac{u}{2}\right)^n$ for $n \geq 1$ (noting $u \geq 0$).

Hence from (13) and (14) we have:

$$\begin{aligned} \frac{u^3}{e^u - 1} &= \frac{u^2}{1 + \frac{u}{2!} + \ldots} \\ &\leq u^2 e^{-\frac{u}{2}} \end{aligned} \tag{15}$$

Convergence is now a heartbeat away since we have:

$$\begin{aligned} \int_0^\infty \frac{u^3}{e^u - 1}\,du &\leq \int_0^\infty u^2 e^{-\frac{u}{2}}\,du \\ &= 2\int_0^\infty (2t)^2 e^{-t}\,dt \\ &= 8\,\Gamma(3) \\ &= 16 \end{aligned} \tag{16}$$

Recall that $\Gamma(u+1) = \int_0^\infty t^u\, e^{-t}\,dt$ and for n positive and integral, $\Gamma(n+1) = n!$

This ought to satisfy the most anally retentive mathematician!

To actually perform the integration requires some analysis. First of all we get by long division:

$$\begin{aligned}\int_0^\infty \frac{u^3}{e^u-1}\,du &= \int_0^\infty u^3(e^{-u}+e^{-2u}+e^{-3u}+\dots)\,du \\ &= \int_0^\infty \sum_{k=1}^\infty u^3\,e^{-ku}\,du\end{aligned} \tag{17}$$

Now (17) screams interchange of integration and summation since we will then have a sum of Gamma functions. For instance, by making the substitution $t = ku$ we get:

$$\begin{aligned}\int_0^\infty u^3 e^{-u}\,du =& \Gamma(4) \\ \int_0^\infty u^3 e^{-2u}\,du =& \frac{1}{16}\int_0^\infty u^3 e^{-u}\,du = \frac{1}{16}\Gamma(4) = \frac{1}{2^4}\Gamma(4) \\ \int_0^\infty u^3 e^{-3u}\,du =& \frac{1}{3^4}\int_0^\infty u^3 e^{-u}\,du = \frac{1}{3^4}\Gamma(4) \\ &\vdots \\ \int_0^\infty u^3 e^{-ku}\,du =& \frac{1}{k^4}\int_0^\infty u^3 e^{-u}\,du = \frac{1}{k^4}\Gamma(4) \\ &\vdots\end{aligned} \tag{18}$$

To interchange the order or of integration and summation we need to establish that for every $u \geq 0$ this infinite series is uniformly convergent:

$$\sum_{k=1}^\infty u^3 e^{-ku} \tag{19}$$

The Weierstrass M-test is the key to establishing this and we need to find a sequence of dominating constants, independent of u, M_k, such that $\sum_{k=1}^\infty M_k$ converges. To this end we note that:

$$\begin{aligned}u^3\,e^{-ku} =& \frac{u^3}{1+ku+\frac{(ku)^2}{2!}+\frac{(ku)^3}{3!}+\dots} \\ \leq& \frac{6}{k^3}\end{aligned} \tag{20}$$

Hence:

$$\begin{aligned}\sum_{k=1}^\infty u^3\,e^{-ku} \leq& 6\sum_{k=1}^\infty \frac{1}{k^3} \\ =& 6\,\zeta(3)\end{aligned} \tag{21}$$

where ζ is the Zeta function.

On this basis we can interchange the order of integration and summation in (17), yielding:

$$\begin{aligned}\int_0^\infty \frac{u^3}{e^u - 1}\, du =& \int_0^\infty \sum_{k=1}^\infty u^3\, e^{-ku}\, du \\ =& \sum_{k=1}^\infty \int_0^\infty u^3\, e^{-ku}\, du \\ =& \Gamma(4) \sum_{k=1}^\infty \frac{1}{k^4}\end{aligned} \tag{22}$$

For those who know something about the sums of p-series, you will know that $\sum_{k=1}^\infty \frac{1}{k^4} = \frac{\pi^4}{90}$, but how do you prove that?

Thus (12) can be finally calcuated, but before we do that we have to prove one more thing.

3 Proof that $\sum_{k=1}^\infty \frac{1}{k^4} = \frac{\pi^4}{90}$ using Fourier theory

Being a Fourier theory fan, I am biased towards such a proof, but in reality it is actually a straightforward way of proving such a result. After all, Parseval's Theorem is regularly used to prove Euler's famous Basel Problem result $\sum_{k=1}^\infty \frac{1}{k^2} = \frac{\pi^2}{6}$.

Recall that the Fourier series for a function $f(x)$ is:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^\infty (a_n \cos nx + b_n \sin nx) \tag{23}$$

where:

$$\begin{aligned} a_0 =& \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx \\ a_n =& \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx \quad n = 1, 2, \ldots \\ b_n =& \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx \quad n = 1, 2, \ldots \end{aligned} \tag{24}$$

If we choose an even function such as $f(x) = x^4$ then the sin terms disappear and we get the Fourier cosine series When we do this note that due to the evenness of $x^4 \cos nx$ we have that $a_n = \frac{2}{\pi} \int_0^\pi x^4 \cos nx\, dx$ for $n = 0, 1, 2, \ldots$. So let's see where that takes us:

$$
\begin{aligned}
x^4 =& \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(\frac{2}{\pi} \int_0^{\pi} x^4 \, \cos nx \, dx \right) \cos nx \\
=& \frac{2}{2\pi} \int_0^{\pi} x^4 \, dx + \sum_{n=1}^{\infty} \left(\frac{2}{\pi} \int_0^{\pi} x^4 \, \cos nx \, dx \right) \cos nx \\
=& \frac{\pi^4}{5} + \sum_{n=1}^{\infty} \left(\frac{2}{\pi} \int_0^{\pi} x^4 \, \cos nx \, dx \right) \cos nx
\end{aligned} \tag{25}
$$

We now need to do a straightforward but typically tedious integration by parts as follows:

$$
\begin{aligned}
\frac{2}{\pi} \int_0^{\pi} x^4 \, \cos nx \, dx =& \frac{2}{\pi} \int_0^{\pi} x^4 \, d\big(\frac{1}{n} \sin nx\big) \\
=& \frac{-8}{n\pi} \int_0^{\pi} x^3 \, \sin nx \, dx \\
=& \frac{-8}{n\pi} \int_0^{\pi} x^3 \, d\Big(\frac{-1}{n} \cos nx\Big) \, dx \\
=& \frac{-8}{n\pi} \Big[\frac{-x^3 \, \cos nx}{n} \Big|_0^{\pi} + \frac{1}{n} \int_0^{\pi} 3x^2 \, \cos nx \, dx \Big] \\
=& \frac{8\pi^2}{n^2} \, \cos n\pi - \frac{24}{n^2\pi} \int_0^{\pi} x^2 \, \cos nx \, dx \\
=& \frac{8\pi^2}{n^2} \, \cos n\pi - \frac{24}{n^2\pi} \int_0^{\pi} x^2 \, d\big(\frac{1}{n} \sin nx\big) \, dx \\
=& \frac{8\pi^2}{n^2} \, \cos n\pi - \frac{24}{n^2\pi} \Big[\frac{-2}{n} \int_0^{\pi} x \, \sin nx \, dx \Big] \\
=& \frac{8\pi^2}{n^2} \, \cos n\pi + \frac{48}{n^3\pi} \int_0^{\pi} x \, \sin nx \, dx \\
=& \frac{8\pi^2}{n^2} \, \cos n\pi + \frac{48}{n^3\pi} \int_0^{\pi} x \, d\big(\frac{-1}{n} \cos nx\big) \, dx \\
=& \frac{8\pi^2}{n^2} \, \cos n\pi + \frac{48}{n^3\pi} \Big[\frac{-x \cos nx}{n} \Big|_0^{\pi} + \underbrace{\frac{1}{n} \int_0^{\pi} \cos nx \, dx}_{=0} \Big] \\
=& \frac{8\pi^2}{n^2} \, \cos n\pi + \frac{48}{n^4\pi} \times -\pi \, \cos n\pi \\
=& \Big(\frac{8\pi^2 n^2 - 48}{n^4} \Big) \, \cos n\pi
\end{aligned} \tag{26}
$$

We now plug the final line of (26) into the final line of (25):

$$
\begin{aligned}
x^4 =& \frac{\pi^4}{5} + \sum_{n=1}^{\infty} \left(\frac{8\pi^2 n^2 - 48}{n^4} \right) \cos n\pi \, \cos nx \\
\pi^4 =& \frac{\pi^4}{5} + \sum_{n=1}^{\infty} \left(\frac{8\pi^2 n^2 - 48}{n^4} \right) \underbrace{\cos^2 n\pi}_{=(-1)^{2n}=1} \\
\pi^4 =& \frac{\pi^4}{5} + 8\pi^2 \sum_{n=1}^{\infty} \frac{1}{n^2} - 48 \sum_{n=1}^{\infty} \frac{1}{n^4} \\
\pi^4 =& \frac{\pi^4}{5} + 8\pi^2 \times \frac{\pi^2}{6} - 48 \sum_{n=1}^{\infty} \frac{1}{n^4} \\
\pi^4 =& \frac{\pi^4}{5} + \frac{4\pi^4}{3} - 48 \sum_{n=1}^{\infty} \frac{1}{n^4} \\
\therefore \sum_{n=1}^{\infty} \frac{1}{n^4} =& \frac{\pi^4}{90}
\end{aligned}
\tag{27}
$$

Where we have used the fact that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ (Euler's Basel Problem). Going back to (22) we therefore have:

$$
\begin{aligned}
\int_0^{\infty} \frac{u^3}{e^u - 1} \, du =& \Gamma(4) \sum_{k=1}^{\infty} \frac{1}{k^4} \\
=& 3! \times \frac{\pi^4}{90} \\
=& \frac{\pi^4}{15}
\end{aligned}
\tag{28}
$$

We now can solve (12) where we make the substitution $u = \frac{h\nu}{kT}$:

$$
\begin{aligned}
\frac{P}{A} =& \frac{2\pi k^3 T^3}{h^2 c^2} \int_0^{\infty} \frac{\left(\frac{h\nu}{kT}\right)^3}{e^{\left(\frac{h\nu}{kT}\right)} - 1} \, d\nu \\
=& \frac{2\pi k^3 T^3}{h^2 c^2} \int_0^{\infty} \frac{u^3}{e^u - 1} \frac{kT}{h} \, du \\
=& \frac{2\pi k^4 T^4}{h^3 c^2} \frac{\pi^4}{15} \\
=& \frac{2\pi^5 k^4}{15 h^3 c^2} T^4 \\
=& \sigma T^4
\end{aligned}
\tag{29}
$$

4 Historical comments

When Planck derived his radiation law in 1900, Lebesgue had not published his novel theory of integration. Fourier theory was well known among physicists (for instance Lord Kelvin had "devoured" Fourier's treatise on heat when he was an adolescent) so the steps in (25)-(27) would not have troubled them and they would have been aware of the fact that convergence was not an issue. The conditions allowing exchange of integration and summation were well known among top physicists. From a Lebesgue integration perspective the interchange of integration and summation is perfectly valid and is guaranteed by some very general theorems - see [3] Chapter 6.

5 References

[1] Max Jammer, "Conceptual Development of Quantum Mechanics", The History of Modern Physics 1800-1950, Volume 12, Tomash Publishers, 1989. The book can be downloaded here: `https://b-ok.cc/book/3706316/7c4073`

[2] `http://scienceworld.wolfram.com/physics/Stefan-BoltzmannLaw.html`

[3] Frank Jones, "Lebesgue Integration on Euclidean Space", Revised Edition,Jones and Bartlett, 2001.

6 History

Created 18 February 2020
23 February 2020 - corrected typo in (27) $(-1)^{2n} = 1$ instead of $(-1)^2 = 1$