Exponential waiting times

A standard probability problem is to determine the expected waiting time for a discrete model where changes can only occur at the following times: δ, 2δ,3δ,... The simplest candidate for a waiting time T is the waiting time for the first success in a sequence of Bernoulli trials with probability p (which conceptually depends on δ). The probability that T > nδ is a string of n failures with probability so P $\{T > n\delta\} = (1-p)^n$. See *William Feller, "An Introduction to Probability and Its Applications", Volume II, Second Edition, Wiley, 1972, pages 1-2*.

What is the expected waiting time E(T)?

This is one of those "foundation" type of problems so it worth doing in detail. The basic definition of expectation leads us to this: $E(T) = \sum_{k=1}^{\infty} P\{T = k\delta\}\, k\delta$

To find P (T = kδ) there is a standard trick which is to note that one can split the set of events into two mutually exclusive sets as follows:

$\{T > (k-1)\delta\} = \{T = k\delta\} \bigcup \{T > k\delta\}$

Hence $P\{T = k\delta\} = P\{T > (k-1)\delta\} - \{T > k\delta\}$

$= (1-p)^{k-1} - (1-p)^k$

$= (1-p)^{k-1}\,[1 - (1-p)]$

$= p(1-p)^{k-1}$

Hence $E(T) = \sum_{k=1}^{\infty} P\{T = k\delta\}\, k\delta$

$= \sum_{k=1}^{\infty} p\,(1-p)^{k-1}\, k\delta$

$= p\delta \sum_{k=1}^{\infty} p\,(1-p)^{k-1}$

Let $S = \sum_{k=1}^{\infty} p\,(1-p)^{k-1} = 1 + 2u + 3u^2 + 4u^3 + \ldots$ where u = 1-p

So $uS = u + 2u^2 + 3u^3 + 4u^4 + \ldots\ldots$

Hence $(1-u)S = 1 + u + u^2 + u^3 + \ldots$

$= \frac{1}{1-u}$ (using the same technique for the sum on the RHS)

Therefore $S = \frac{1}{(1-u)^2} = \frac{1}{p^2}$

Finally, $E(T) = p\delta.\ \frac{1}{p^2} = \frac{\delta}{p}$

Now if as δ becomes smaller in such a way that the expectation $E(T) = \frac{\delta}{p} = \alpha$ is fixed and we take some period of time t there will correspondingly be n trials where n is approximately $\frac{t}{\delta}$ (remember that the expected waiting time was based on $T > n\delta$) . For small δ we can say that $P\{T > t\} = (1-p)^n \approx \left(1 - \frac{\delta}{\alpha}\right)^{\frac{t}{\delta}} \approx e^{\frac{-t}{\alpha}}$ Note here that since we have assumed $E(T) = \frac{\delta}{p} = \alpha$ then $p = \frac{\delta}{\alpha}$. If you can't see how $e^{\frac{-t}{\alpha}}$ was obtained consider these steps:

$e^x = \lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n$ and let $x = -\frac{\delta n}{\alpha}$ so that:

$(1-p)^n \approx \left(1 - \frac{\delta}{\alpha}\right)^n = \left(1 - \frac{\frac{\delta n}{\alpha}}{n}\right)^n \approx e^{\frac{-t}{\alpha}}$ (since $n \approx \frac{t}{\delta}$, $-\frac{\delta n}{\alpha} \approx \frac{-t}{\alpha}$)

Thus when the waiting time is considered as a geometrically distributed random variable one gets an exponential distribution. Feller points out it would be more "intuitive" to start with a sample space of real numbers and then introduce the exponential distribution directly. The above argument takes a granular (discrete) approach and a limiting process to get to the exponential distribution.

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