Extracting the Gaussian from integrals of cosines

Peter Haggstrom
mathsatbondibeach@gmail.com
https://gotohaggstrom.com

February 6, 2020

1 Background

At first blush it seems unlikely that the Gaussian could be somehow conjured into existence from the cosine function, but Paul Lévy does precisely that in his book *Lecons d’Analyse Functionnelle* ([1], pages 261-265) in the context of deriving expressions for the volume of a sphere in \( n \) Euclidean dimensions. Norbert Wiener specifically credits Lévy for the inspiration to apply Lévy’s approach to infinitely many dimensions in the context of Wiener’s development of Brownian motion. ([2], page 132).

Lévy deduces the volume of an \( n \) dimensional sphere from the following integral:

\[
I_n = \int_0^{\pi/2} \cos^n \theta \, d\theta
\]  

A straightforward integration by parts gives:

\[
I_n = \left[ \sin \theta \cos^{n-1} \theta \right]_0^{\pi/2} + (n-1) \int_0^{\pi/2} \cos^{n-2} \theta \, \sin^2 \theta \, d\theta
\]

\[
= (n-1) \int_0^{\pi/2} \frac{1}{2} \cos^n \theta \, (1 - \cos^2 \theta) \, d\theta
\]

\[
= (n-1) (I_{n-2} - I_n)
\]

Therefore we get the recurrence relation:

\[
I_n = \frac{n-1}{n} I_{n-2}
\]

To develop the formula for general \( n \) we start with \( I_1 \) and \( I_2 \):

\[
I_1 = \int_0^{\pi/2} \cos \theta \, d\theta = \left[ \sin \theta \right]_0^{\pi/2} = 1
\]

\[
I_2 = \int_0^{\pi/2} \cos^2 \theta \, d\theta = \int_0^{\pi/2} \frac{1}{2} (\cos 2\theta + 1) \, d\theta = \frac{1}{2} \left[ \frac{1}{2} \sin 2\theta + \theta \right]_0^{\pi/2} = \frac{\pi}{4}
\]
We now calculate $I_{2p}$ and $I_{2p+1}$ separately (using (4) and (5)):

\[
I_{2p} = \frac{2p - 1}{2p} I_{2p-2} = \left(\frac{2p - 1}{2p}\right) \left(\frac{2p - 3}{2p - 2}\right) I_{2p-4} = \cdots \tag{6}
\]

\[
= \frac{1}{2p(2p-2)(2p-4)\ldots 4} \frac{(2p-1)(2p-3)(2p-5)\ldots 3}{2p(2p+1)(2p-1)(2p-3)\ldots 5} \pi
\]

Similarly:

\[
I_{2p+1} = \frac{2p + 1}{2p + 2} I_{2p-1} = \left(\frac{2p}{2p+1}\right) \left(\frac{2p - 2}{2p - 1}\right) I_{2p-3} = \cdots \tag{7}
\]

\[
= \frac{2p(2p-2)(2p-4)\ldots 4}{(2p + 1)(2p-1)(2p-3)\ldots 5} \pi
\]

It follows from (6) and (7) that:

\[
I_{2p}I_{2p+1} = \frac{1}{2p+1} \frac{\pi}{2} \tag{8}
\]

and

\[
I_{2p-1}I_{2p} = \frac{1}{2p} \frac{\pi}{2} \tag{9}
\]

[Note there is a typo in (1), page 263 where Lévy has $I_{2p+1}I_{2p} = \frac{1}{2p} \frac{\pi}{2}$]

It follows that for any positive integer $n$:

\[
I_nI_{n-1} = \frac{\pi}{2n} \tag{10}
\]
For large $n$, $I_n$ and $I_{n-1}$ are comparable so that (10) implies that:

$$I_n \sim \sqrt{\frac{\pi}{2n}}$$  \hspace{1cm} (11)

The integral reduction formula (3) essentially forms the basis for how Wallis’s product formula is derived in calculus courses (although $\sin^n x$ is used).

Note that since for $x \in (0, \frac{\pi}{2})$, $\cos^{n+2} x < \cos^{n+1} x < \cos^n x$ for all $n \geq 1$, it follows that:

$$I_n > I_{n+1} > I_{n+2}$$  \hspace{1cm} (12)

Now for $\alpha \in (0, \frac{\pi}{2})$ we have:

$$\int_{\alpha}^{\frac{\pi}{2}} \cos^n \theta d\theta < (\frac{\pi}{2} - \alpha) \cos^n \alpha$$  \hspace{1cm} (13)

[Note that for $\alpha, \theta \in (0, \frac{\pi}{2})$, $\cos \alpha > \cos \theta$ for $\alpha < \theta$]

Thus as $n \to \infty$, $\int_{\alpha}^{\frac{\pi}{2}} \cos^n \theta d\theta \to 0$ since $\cos^n \alpha \to 0$.

Because $I_n = \int_0^\alpha \cos^n \theta d\theta + \int_{\alpha}^{\frac{\pi}{2}} \cos^n \theta d\theta$ this means that $I_n$ can be approximated by $\int_0^\alpha \cos^n \theta d\theta$ for large $n$. However, Lévy notes that to have a finite fraction of $I_n$ one must have an upper limit on the integral which approaches zero with $n$. That this is so can be seen from the following graph of $\int_0^\alpha \cos^{1000} x \, dx$:

What this graph demonstrates is that for large $n$, values of $\alpha$ not much greater than zero give rise to an integral $\int_0^\alpha \cos^n x \, dx$ which quickly approximates $I_n$, hence to get what Lévy calls a ”finite fraction of $I_n$” ( [1], page 264) one must consider values of $\alpha$
very close to zero. Hence he chooses an upper limit of the form \( \frac{\alpha}{\sqrt{n}} \) which approaches zero as \( n \to \infty \). By substitution, this leads to:

\[
\int_{0}^{\frac{\alpha}{\sqrt{n}}} \cos^n \theta \, d\theta = \frac{1}{\sqrt{n}} \int_{0}^{\alpha} \cos^n \frac{x}{\sqrt{n}} \, dx = \frac{1}{\sqrt{n}} \int_{0}^{\alpha} e^{-\frac{x^2}{2}} \, dx
\]

Lévy says that for \( x \in [0, \alpha] \), \( \cos^n \frac{x}{\sqrt{n}} \) approaches \( e^{-\frac{x^2}{2}} \) uniformly as \( n \to \infty \). Here is what the approximation looks like for \( n = 10 \):

![Graph showing the comparison between \( \cos^{10} \left( \frac{x}{\sqrt{10}} \right) \) and \( e^{-\frac{x^2}{2}} \).]

The two functions are hard to tell apart from the graph.

The limiting behaviour is justified as follows:

\[
cos^n \frac{x}{\sqrt{n}} = \left( 1 - 2 \sin^2 \frac{x}{2\sqrt{n}} \right)^n
\]

\[
= \left( 1 - 2 \left( \frac{x}{2\sqrt{n}} \right)^2 \frac{x^2}{4n} \right)^n
\]

\[
= \left[ 1 - \left( \frac{\sin \frac{x}{2\sqrt{n}}}{\frac{x}{2\sqrt{n}}} \right)^2 \frac{x^2}{2n} \right]^n
\]

But for fixed \( x \), \( \frac{\sin \frac{x}{2\sqrt{n}}}{\frac{x}{2\sqrt{n}}} \to 1 \) as \( n \to \infty \), hence \( \left( 1 - \left( \frac{\sin \frac{x}{2\sqrt{n}}}{\frac{x}{2\sqrt{n}}} \right)^2 \frac{x^2}{2n} \right)^n \to e^{-\frac{x^2}{2}} \). That the convergence is uniform is suggested by this diagram below (which is for \( n = 20 \)).
Analytically, the convergence is uniform because, for any \( \epsilon > 0 \) we can find an \( N(\epsilon) \) (to emphasise that \( N \) depends on \( \epsilon \)) such that for all \( n > N(\epsilon) \) and for all \( x \in [0, \alpha] \) we have:

\[
\left| \cos^n \frac{x}{\sqrt{n}} - e^{-\frac{x^2}{2}} \right| < \epsilon
\]

We have that:

\[
\left| \cos^n \frac{x}{\sqrt{n}} - e^{-\frac{x^2}{2}} \right| = \left| 1 - \left( \frac{\sin \frac{x}{2\sqrt{n}}}{\frac{x}{2\sqrt{n}}} \right)^2 \frac{x^2}{2n} - e^{-\frac{x^2}{2}} \right|
\]

\[
\leq \left| e^{-\left( \frac{\sin \frac{x}{2\sqrt{n}}}{\frac{x}{2\sqrt{n}}} \right)^2 \frac{x^2}{2}} - e^{-\frac{x^2}{2}} \right|
\]

\[
= \left| e^{\left( 1 - \left( \frac{\sin \frac{x}{2\sqrt{n}}}{\frac{x}{2\sqrt{n}}} \right)^2 \right) \frac{x^2}{2}} - 1 \right|
\]

\[
\leq \left| e^{\left( 1 - \left( \frac{\sin \frac{x}{2\sqrt{n}}}{\frac{x}{2\sqrt{n}}} \right)^2 \right) \frac{x^2}{2}} - 1 \right|
\]

Because \( \frac{\sin \frac{x}{2\sqrt{n}}}{\frac{x}{2\sqrt{n}}} \) is an increasing sequence which converges to 1 we can find an \( N(\epsilon) \) sufficiently large so that \( 1 - \frac{\sin \frac{x}{2\sqrt{n}}}{\frac{x}{2\sqrt{n}}} < \epsilon' \) for all \( n > N(\epsilon') \). Thus:

\[
1 - \left( \frac{\sin \frac{x}{2\sqrt{n}}}{\frac{x}{2\sqrt{n}}} \right)^2 = \left( 1 - \frac{\sin \frac{x}{2\sqrt{n}}}{\frac{x}{2\sqrt{n}}} \right) \left( 1 + \frac{\sin \frac{x}{2\sqrt{n}}}{\frac{x}{2\sqrt{n}}} \right) < 2\epsilon'
\]
since $\frac{\sin \frac{x}{\sqrt{n}}}{\sqrt{n}} \leq 1$ for all $x \in [0, \alpha) \subset [0, \frac{\pi}{2}]$ and all $n$.

Using (18) we can therefore dominate the RHS of (17) as follows:

$$
\left| e^{\left[1 - \left(\frac{\sin \frac{x}{\sqrt{n}}}{\sqrt{n}}\right)^2\right]} - 1 \right| < e^{\epsilon'} - 1 \leq e^{\frac{\epsilon^2}{4}} - 1 < e^{\frac{\epsilon'}{2}} - 1 \quad \text{(19)}
$$

Thus recalling (16), we need $e^{\frac{\epsilon'}{2}} < 1 + \epsilon$ so we need $\epsilon' < \frac{2}{\epsilon^2} \ln(1 + \epsilon)$. Suppose we set $\epsilon = 0.001$ then we need to find $N(\epsilon')$ such that $1 - \frac{\sin \frac{x}{\sqrt{n}}}{\sqrt{n}} < \epsilon'$ for all $n > N(\epsilon')$.

Since $\epsilon' = 0.003998$ Mathematica tells us that $N = 258$ will ensure that $1 - \frac{\sin \frac{x}{\sqrt{n}}}{\sqrt{n}} < 0.0003998$. The graph below visually demonstrates that $N = 258$ is the threshold value:

Thus for $n > 258$ we can ensure that $\left| \cos^n \frac{x}{\sqrt{n}} - e^{-\frac{x^2}{2}} \right| < 0.001$ for all $x \in [0, \frac{\pi}{2}]$.

Taking $x = 0$, $x = 1$ and $x = \frac{\pi}{2}$, as an example, Mathematica gives the following values for $\Delta[x, n] = \left| \cos^n \frac{x}{\sqrt{n}} - e^{-\frac{x^2}{2}} \right|$:

- $\Delta[0, 259] = 0 < 0.001$
- $\Delta[1, 259] = 0.0001953 < 0.001$
- $\Delta[\frac{\pi}{2}, 259] = 0.0005713 < 0.001$
- $\Delta[1, 2000] = 0.000025275 < 0.001$
- $\Delta[\frac{\pi}{2}, 2000] = 0.000073886 < 0.001$

So the convergence that looked uniform is uniform.

Lévy uses this basic result in the context of deriving formulas for the volumes of $n$ dimensional spheres and then applying that to determining certain probabilities - hence the link with Wiener’s Brownian motion work.
2 References

[1] Paul Lévy, Lecons d’Analyse Functionnelle, Gauthier-Villars, Paris, 1922. There is no English translation of this book as far as I know but the French version can be accessed here: https://archive.org/details/leconsdanalysefo00levyrich


3 History

Created 06 February 2020