Extracting the Gaussian from integrals of cosines

Peter Haggstrom
mathsatbondibeach@gmail.com
https://gotohaggstrom.com

February 6, 2020

1 Background

At first blush it seems unlikely that the Gaussian could be somehow conjured into existence from the cosine function, but Paul Lévy does precisely that in his book *Lecons d'Analyse Functionnelle* ([1], pages 261-265) in the context of deriving expressions for the volume of a sphere in n Euclidean dimensions. Norbert Wiener specifically credits Lévy for the inspiration to apply Lévy's approach to infinitely many dimensions in the context of Wiener's development of Brownian motion. ([2], page 132).

Lévy deduces the volume of an n dimensional sphere from the following integral:

$$I_n = \int_0^{\frac{\pi}{2}} \cos^n \theta \, d\theta \tag{1}$$

A straightforward integration by parts gives:

$$\begin{aligned} I_n =& \Big[\sin\theta \, \cos^{n-1}\theta \Big]_0^{\frac{\pi}{2}} + (n-1) \int_0^{\frac{\pi}{2}} \cos^{n-2}\theta \, \sin^2\theta \, d\theta \\ =& (n-1) \int_0^{\frac{\pi}{2}} \cos^{n-2}\theta \, (1 - \cos^2\theta) \, d\theta \\ =& (n-1)(I_{n-2} - I_n) \end{aligned} \tag{2}$$

Therefore we get the recurrence relation:

$$I_n = \frac{n-1}{n} \, I_{n-2} \tag{3}$$

To develop the formula for general n we start with I_1 and I_2:

$$I_1 = \int_0^{\frac{\pi}{2}} \cos\theta \, d\theta = \Big[\sin\theta \Big]_0^{\frac{\pi}{2}} = 1 \tag{4}$$

$$I_2 = \int_0^{\frac{\pi}{2}} \cos^2\theta \, d\theta = \int_0^{\frac{\pi}{2}} \frac{1}{2}(\cos 2\theta + 1) \, d\theta = \frac{1}{2}\Big[\frac{1}{2}\sin 2\theta + \theta \Big]_0^{\frac{\pi}{2}} = \frac{\pi}{4} \tag{5}$$

We now calculate I_{2p} and I_{2p+1} separately (using (4) and (5)):

$$\begin{aligned}
I_{2p} =& \frac{2p-1}{2p} I_{2p-2} \\
=& \left(\frac{2p-1}{2p}\right)\left(\frac{2p-3}{2p-2}\right) I_{2p-4} \\
=& \left(\frac{2p-1}{2p}\right)\left(\frac{2p-3}{2p-2}\right)\left(\frac{2p-5}{2p-4}\right) I_{2p-6} \\
=& \dots \\
=& \left(\frac{2p-1}{2p}\right)\left(\frac{2p-3}{2p-2}\right)\left(\frac{2p-5}{2p-4}\right) \dots \left(\frac{3}{4}\right) I_2 \\
=& \frac{(2p-1)(2p-3)(2p-5)\dots 3}{2p(2p-2)(2p-4)\dots 4} \frac{\pi}{4} \\
=& \frac{(2p-1)(2p-3)(2p-5)\dots 3.1}{2p(2p-2)(2p-4)\dots 4.2} \frac{\pi}{2}
\end{aligned} \tag{6}$$

Similarly:

$$\begin{aligned}
I_{2p+1} =& \frac{2p}{2p+1} I_{2p-1} \\
=& \left(\frac{2p}{2p+1}\right)\left(\frac{2p-2}{2p-1}\right) I_{2p-3} \\
=& \left(\frac{2p}{2p+1}\right)\left(\frac{2p-2}{2p-1}\right)\left(\frac{2p-4}{2p-3}\right) I_{2p-5} \\
=& \dots \\
=& \left(\frac{2p}{2p+1}\right)\left(\frac{2p-2}{2p-1}\right)\left(\frac{2p-4}{2p-3}\right) \dots \left(\frac{4}{5}\right)\left(\frac{2}{3}\right) I_1 \\
=& \frac{(2p)(2p-2)(2p-4)\dots 4.2}{(2p+1)(2p-1)(2p-3)\dots 5.3}
\end{aligned} \tag{7}$$

It follows from (6) and (7) that:

$$I_{2p} I_{2p+1} = \frac{1}{2p+1} \frac{\pi}{2} \tag{8}$$

and

$$I_{2p-1} I_{2p} = \frac{1}{2p} \frac{\pi}{2} \tag{9}$$

[Note there is a typo in ([1], page 263 where Lévy has $I_{2p+1} I_{2p} = \frac{1}{2p} \frac{\pi}{2}$]
It follows that for any positive integer n:

$$I_n I_{n-1} = \frac{\pi}{2n} \tag{10}$$

For large n, I_n and I_{n-1} are comparable so that (10) implies that:

$$I_n \sim \sqrt{\frac{\pi}{2n}} \tag{11}$$

The integral reduction formula (3) essentially forms the basis for how Wallis's product formula is derived in calculus courses (although $\sin^n x$ is used).

Note that since for $x \in (0, \frac{\pi}{2})$, $\cos^{n+2} x < \cos^{n+1} x < \cos^n x$ for all $n \geq 1$, it follows that:

$$I_n > I_{n+1} > I_{n+2} \tag{12}$$

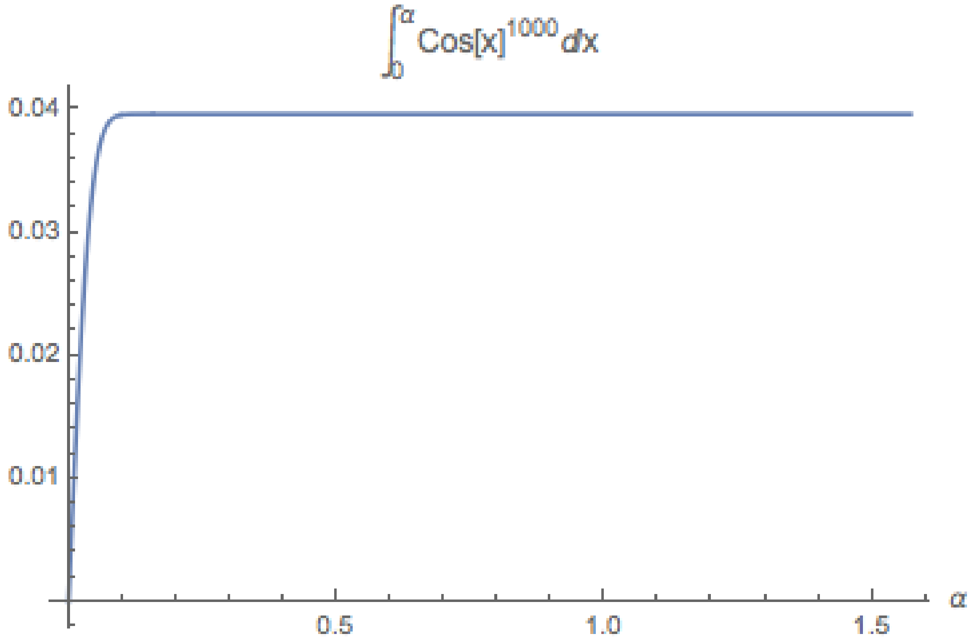
Now for $\alpha \in (0, \frac{\pi}{2})$ we have:

$$\int_{\alpha}^{\frac{\pi}{2}} \cos^n \theta \, d\theta < (\frac{\pi}{2} - \alpha) \cos^n \alpha \tag{13}$$

[Note that for $\alpha, \theta \in (0, \frac{\pi}{2})$, $\cos \alpha > \cos \theta$ for $\alpha < \theta$]

Thus as $n \to \infty$, $\int_{\alpha}^{\frac{\pi}{2}} \cos^n \theta \, d\theta \to 0$ since $\cos^n \alpha \to 0$.

Because $I_n = \int_0^{\alpha} \cos^n \theta \, d\theta + \int_{\alpha}^{\frac{\pi}{2}} \cos^n \theta \, d\theta$ this means that I_n can be approximated by $\int_0^{\alpha} \cos^n \theta \, d\theta$ for large n. However, Lévy notes that to have a finite fraction of I_n one must have an upper limit on the integral which approaches zero with n. That this is so can be seen from the following graph of $\int_0^{\alpha} \cos^{1000} x \, dx$:

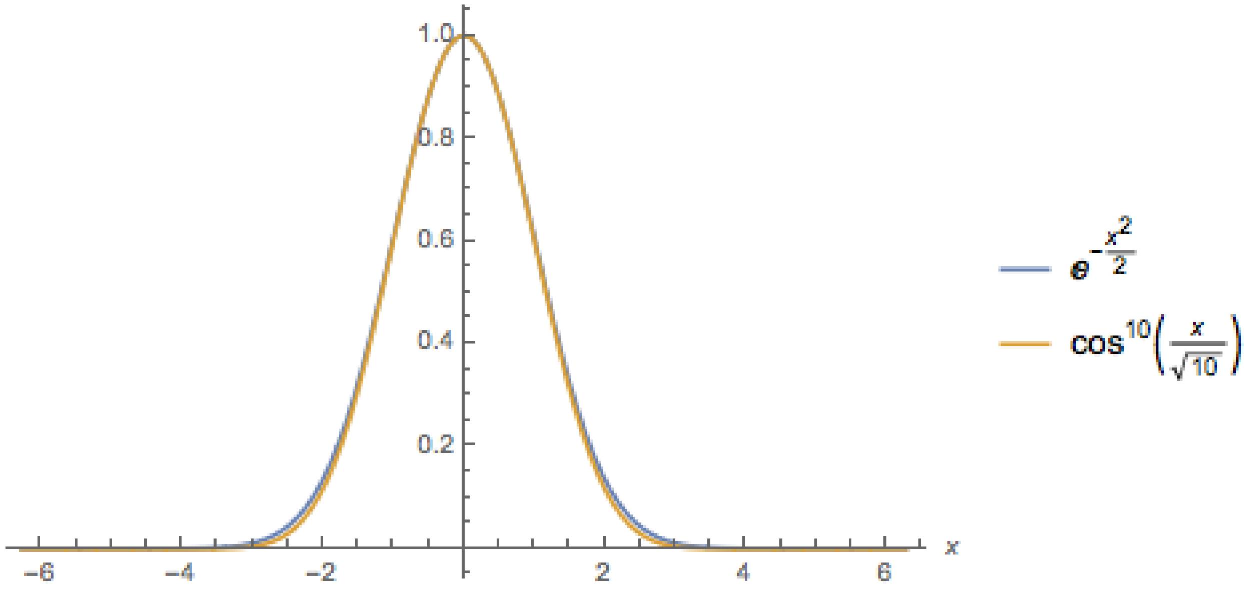


What this graph demonstrates is that for large n, values of α not much greater than zero give rise to an integral $\int_0^{\alpha} \cos^n x \, dx$ which quickly approximates I_n, hence to get what Lévy calls a "finite fraction of I_n" ([1], page 264]) one must consider values of α

very close to zero. Hence he chooses an upper limit of the form $\frac{\alpha}{\sqrt{n}}$ which approaches zero as $n \to \infty$. By substitution, this leads to:

$$\int_0^{\frac{\alpha}{\sqrt{n}}} \cos^n \theta \, d\theta = \frac{1}{\sqrt{n}} \int_0^{\alpha} \cos^n \frac{x}{\sqrt{n}} \, dx = \frac{1}{\sqrt{n}} \int_0^{\alpha} e^{-\frac{x^2}{2}} \, dx \tag{14}$$

Lévy says that for $x \in [0, \alpha]$, $\cos^n \frac{x}{\sqrt{n}}$ approaches $e^{-\frac{x^2}{2}}$ uniformly as $n \to \infty$. Here is what the approximation looks like for $n = 10$:

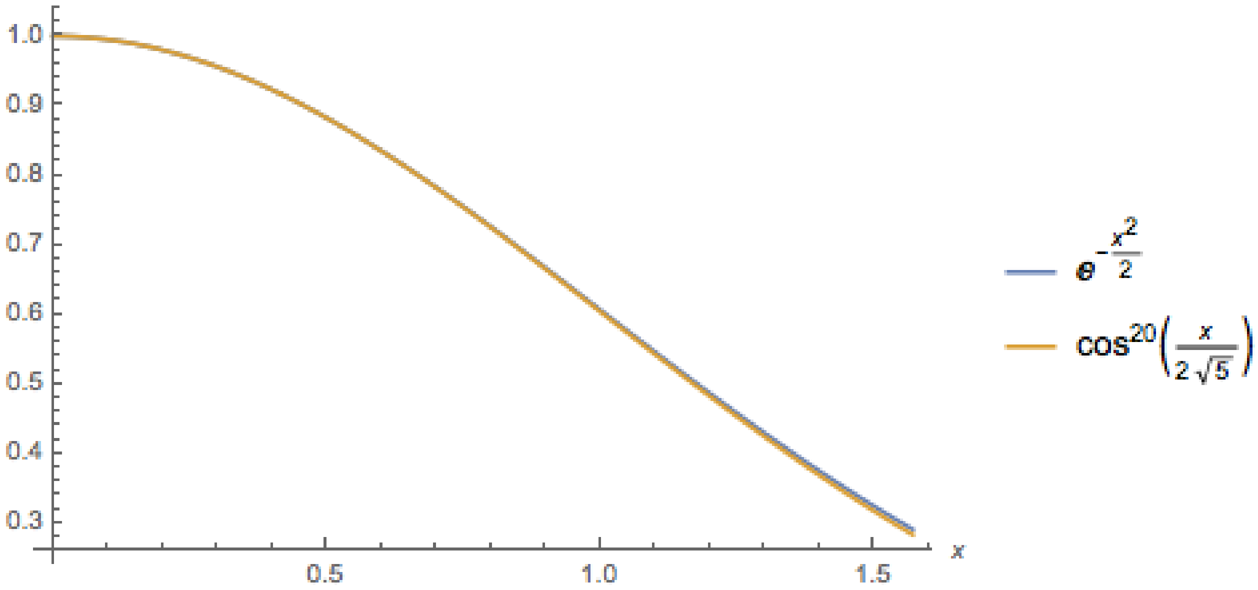


The two functions are hard to tell apart from the graph.

The limiting behaviour is justified as follows:

$$\begin{aligned} \cos^n \frac{x}{\sqrt{n}} &= \Big(1 - 2\sin^2 \frac{x}{2\sqrt{n}}\Big)^n \\ &= \Big(1 - 2\frac{\sin^2 \frac{x}{2\sqrt{n}}}{(\frac{x}{2\sqrt{n}})^2}\, \frac{x^2}{4n}\Big)^n \\ &= \Big[1 - \Big(\frac{\sin \frac{x}{2\sqrt{n}}}{\frac{x}{2\sqrt{n}}}\Big)^2 \frac{x^2}{2n}\Big]^n \end{aligned} \tag{15}$$

But for fixed x, $\frac{\sin \frac{x}{2\sqrt{n}}}{\frac{x}{2\sqrt{n}}} \to 1$ as $n \to \infty$, hence $\Big(1 - \Big(\frac{\sin \frac{x}{2\sqrt{n}}}{\frac{x}{2\sqrt{n}}}\Big)^2 \frac{x^2}{2n}\Big)^n \to e^{\frac{-x^2}{2}}$. That the convergence is uniform is suggested by this diagram below (which is for $n = 20$).



Analytically, the convergence is uniform because, for any $\epsilon > 0$ we can find an $N(\epsilon)$ (to emphasise that N depends on ϵ) such that for all $n > N(\epsilon)$ and for all $x \in [0, \alpha]$ we have:

$$\left|\cos^n \frac{x}{\sqrt{n}} - e^{-\frac{x^2}{2}}\right| < \epsilon \tag{16}$$

We have that :

$$\begin{aligned}\left|\cos^n \frac{x}{\sqrt{n}} - e^{-\frac{x^2}{2}}\right| &= \left|\left[1 - \left(\frac{\sin\frac{x}{2\sqrt{n}}}{\frac{x}{2\sqrt{n}}}\right)^2 \frac{x^2}{2n}\right]^n - e^{-\frac{x^2}{2}}\right| \\ &\leq \left|e^{-\left(\frac{\sin\frac{x}{2\sqrt{n}}}{\frac{x}{2\sqrt{n}}}\right)^2 \frac{x^2}{2}} - e^{-\frac{x^2}{2}}\right| \\ &= \frac{\left|e^{\left[1-\left(\frac{\sin\frac{x}{2\sqrt{n}}}{\frac{x}{2\sqrt{n}}}\right)^2\right]\frac{x^2}{2}} - 1\right|}{e^{\frac{x^2}{2}}} \\ &\leq \left|e^{\left[1-\left(\frac{\sin\frac{x}{2\sqrt{n}}}{\frac{x}{2\sqrt{n}}}\right)^2\right]\frac{x^2}{2}} - 1\right|\end{aligned} \tag{17}$$

Because $\frac{\sin\frac{x}{2\sqrt{n}}}{\frac{x}{2\sqrt{n}}}$ is an increasing sequence which converges to 1 we can find an $N(\epsilon)$ sufficiently large so that $1 - \frac{\sin\frac{x}{2\sqrt{n}}}{\frac{x}{2\sqrt{n}}} < \epsilon^{'}$ for all $n > N(\epsilon^{'})$. Thus:

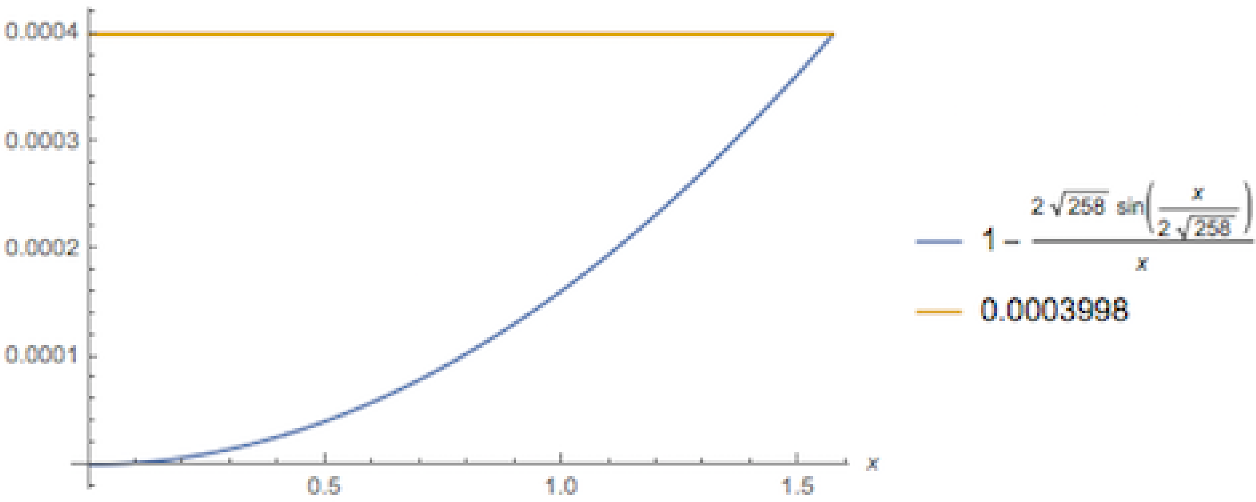
$$1 - \left(\frac{\sin\frac{x}{2\sqrt{n}}}{\frac{x}{2\sqrt{n}}}\right)^2 = \left(1 - \frac{\sin\frac{x}{2\sqrt{n}}}{\frac{x}{2\sqrt{n}}}\right)\left(1 + \frac{\sin\frac{x}{2\sqrt{n}}}{\frac{x}{2\sqrt{n}}}\right) < 2\epsilon^{'} \tag{18}$$

since $\frac{\sin\frac{x}{2\sqrt{n}}}{\frac{x}{2\sqrt{n}}} \leq 1$ for all $x \in [0,\alpha) \subset [0,\frac{\pi}{2}]$ and all n.

Using (18) we can therefore dominate the RHS of (17) as follows:

$$\left|e^{\left[1-\left(\frac{\sin\frac{x}{2\sqrt{n}}}{\frac{x}{2\sqrt{n}}}\right)^2\right]\frac{x^2}{2}} - 1\right| < e^{\epsilon' x^2} - 1 \leq e^{\frac{\pi^2}{4}\epsilon'} - 1 < e^{\frac{5}{2}\epsilon'} - 1 \tag{19}$$

Thus recalling (16), we need $e^{\frac{5}{2}\epsilon'} < 1+\epsilon$ so we need $\epsilon' < \frac{2}{5}\ln(1+\epsilon)$. Suppose we set $\epsilon = 0.001$ then we need to find $N(\epsilon')$ such that $1 - \frac{\sin\frac{x}{2\sqrt{n}}}{\frac{x}{2\sqrt{n}}} < \epsilon'$ for all $n > N(\epsilon')$. Since $\epsilon' = 0.0003998$ Mathematica tells us that $N = 258$ will ensure that $1 - \frac{\sin\frac{x}{2\sqrt{n}}}{\frac{x}{2\sqrt{n}}} <$ 0.0003998. The graph below visually demonstrates that $N = 258$ is the threshold value:



Thus for $n > 258$ we can ensure that $\left|\cos^n\frac{x}{\sqrt{n}} - e^{-\frac{x^2}{2}}\right| < 0.001$ for all $x \in [0,\frac{\pi}{2}]$. Taking $x = 0$, $x = 1$ and $x = \frac{\pi}{2}$, as an example, Mathematica gives the following values for $\Delta[x,n] = \left|\cos^n\frac{x}{\sqrt{n}} - e^{-\frac{x^2}{2}}\right|$:

$\Delta[0,259] = 0 < 0.001$
$\Delta[1,259] = 0.0001953 < 0.001$
$\Delta[\frac{\pi}{2},259] = 0.0005713 < 0.001$
$\Delta[1,2000] = 0.000025275 < 0.001$
$\Delta[\frac{\pi}{2},2000] = 0.000073886 < 0.001$

So the convergence that looked uniform is uniform.

Lévy uses this basic result in the context of deriving formulas for the volumes of n dimensional spheres and then applying that to determining certain probabilities - hence the link with Wiener's Brownian motion work.

2 References

[1] Paul Lévy, Lecons d'Analyse Functionnelle, Gauthier-Villars, Paris, 1922. There is no English translation of this book as far as I know but the French version can be accessed here: `https://archive.org/details/leconsdanalysefo00levyrich`

[2] Norbert Wiener (ed), Armand Siegel (ed), Bayard Rankin (ed), William Ted Martin (ed), "Differential Space, Quantum Spaces, and Prediction", MIT Press, 1966

3 History

Created 06 February 2020