Fundamental properties of exponentials and logarithms

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1 Introduction

Integration theory frequently gives rose to integrals involving exponentials and/or logarithms (and in this paper all references are to logarithms to the base e). Fourier theory is of course a classical example of the involvement of exponentials in integrands. Students studying calculus are often presented with concocted integrals (ie those for which there may be no actual or very little physical basis) which contain the classic mix of exponentials, logs and sine or cosine functions and so they have to work out limits such as these:

$$\lim_{x \to 0} e^{-x} \ln x \sin x \tag{1}$$

and

$$\lim_{x \to \infty} e^{-x} \ln x \sin x \tag{2}$$

There are two properties which are absolutely fundamental in this context:

Property 1:
The logarithm of x tends to infinity with x, but more slowly than any power of x whether that is integral or fractional. So the logarithm gets arbitrarily large but slower than any power. This property is expressed as:

$$\ln x \to \infty \text{ as } x \to \infty \text{ but } \frac{\ln x}{x^{\alpha}} \to 0 \text{ for all positive rational values of } \alpha \tag{3}$$

Property 2:
The function e^y tends to infinity with y more rapidly than any power of y ie as $y \to \infty$, $e^y \to \infty$ but:

$$\lim_{y \to \infty} \frac{y^{\alpha}}{e^y} = 0 \text{ for all values of } \alpha \text{ however great} \tag{4}$$

In many functional analysis applications such as in Fourier theory, this property is used over and over to justify certain limits being zero or getting manageable bounds on functions.

In his book " A Course of Pure Mathematics", ([1] , pages 400-401) Hardy sets out a series of exercises (most of which are from the Tripos exams) which set out some useful analytical properties of $\ln x$. These are listed below:

Property 3:

$$\frac{x}{1+x} < \ln(1+x) < x \text{ for } x > 0 \tag{5}$$

Property 4:

$$x < -\ln(1-x) < \frac{x}{1-x} \text{ for } 0 < x < 1 \tag{6}$$

Property 5:

$$x - \frac{1}{2}x^2 < \ln(1+x) \text{ for } x > 0 \tag{7}$$

Property 6:

$$\frac{x-1}{x} < \ln x < x - 1 \text{ for } x > 1 \tag{8}$$

Property 7:

$$4(x-1) - 2\ln x < 2x\ln x < x^2 - 1 \text{ for } x > 1 \tag{9}$$

Property 8:

$$0 < \frac{1}{x} - \ln\frac{x+1}{x} < \frac{1}{2x^2} \text{ for } x > 0 \tag{10}$$

Property 9:

$$\frac{2}{2x+1} < \ln\frac{x+1}{x} < \frac{2x+1}{2x(x+1)} \text{ for } x > 0 \tag{11}$$

Property 10:

$$\lim_{x\to 1}\frac{\ln x}{x-1} = \lim_{y\to 0}\frac{\ln(1+y)}{y} = 1 \tag{12}$$

The proofs of Properties 3-9 are based on a simple estimation process of the area under the curve $y = \frac{1}{x}$ given that the logarithm can be defined as this:

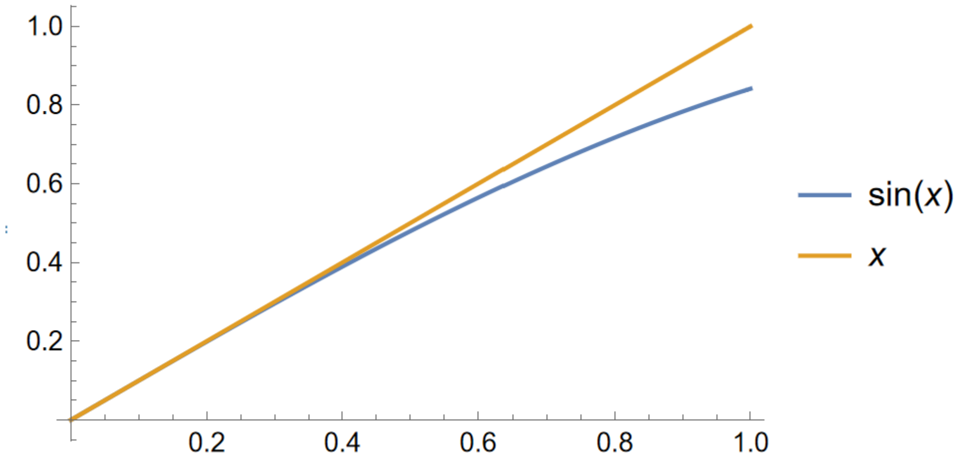
$$\ln x = \int_1^x \frac{dt}{t} \tag{13}$$

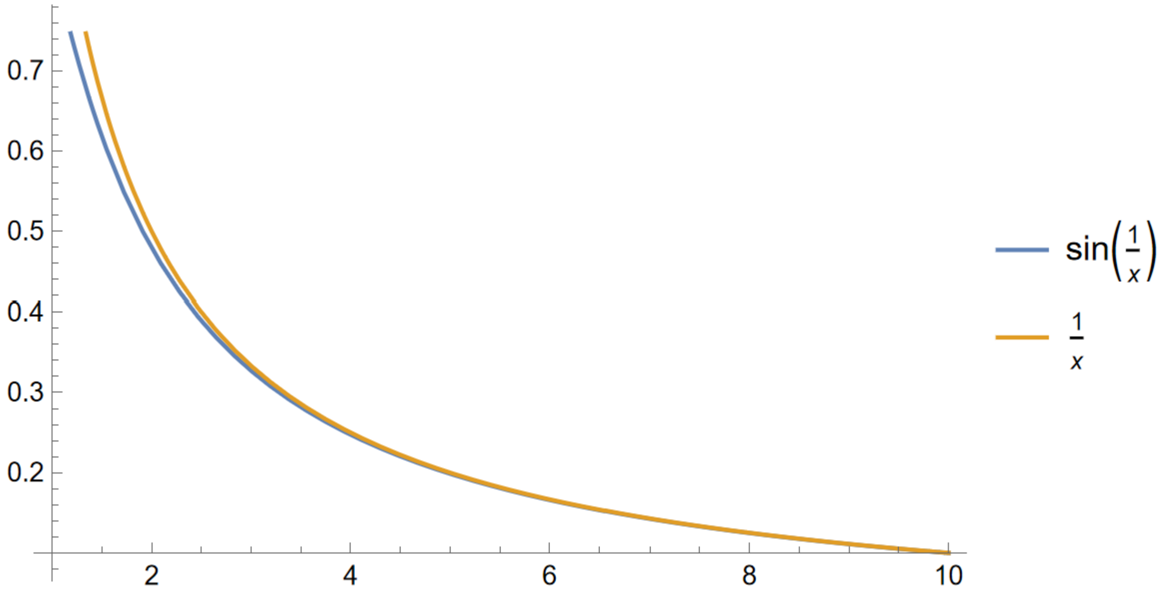
The proofs of all properties will be given below but we will use them to find the limit in (1).

To find the limit of (1) one strategy might be that if you can express the limit as $\lim_{x\to 0} f(x)g(x)$ where $\lim_{x\to 0} f(x) = A$ and $\lim_{x\to 0} g(x) = B$ you would be home and hosed. Property 6 looks like it might be of some help but we need to make the substitution $x = \frac{1}{u}$ where $u > 1$. Then $x > 0$ and as $u \to \infty$, it follows that $x \to 0$. An educated guess at the limit is that it is zero because $\sin x \approx x$ for small x and since $\ln x$ is an infinite sum of powers each of which falls within the scope of Property 1, the ratio $\frac{\ln x}{e^x}$ should approach 0 and so with $f(x) = \frac{\ln x}{e^x}$ and $g(x) = \sin x$ we can use our basic theorem of the limit of a product to get the result. This high level guess must be made rigorous. With our substiution we see that:

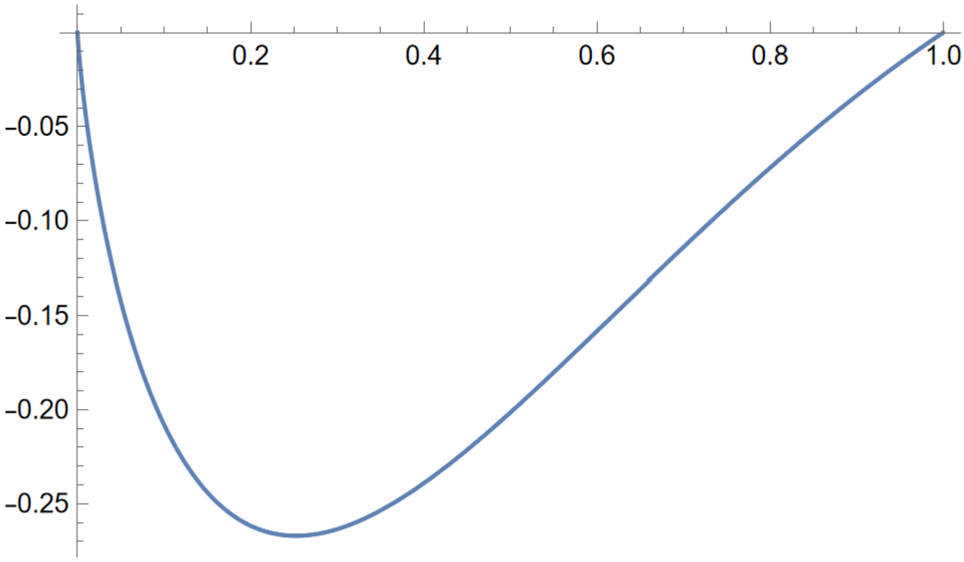
$$\begin{aligned} e^{-x} \ln x \sin x =& e^{-\frac{1}{u}} \ln \frac{1}{u} \sin \frac{1}{u} \\ =& - e^{-\frac{1}{u}} \ln u \sin \frac{1}{u} \\ \le& - e^{-\frac{1}{u}} \frac{\ln u}{u} \end{aligned} \tag{14}$$

But we know that as $u \to \infty, e^{-\frac{1}{u}} \to 1$ and Property 1 tells us that as $u \to \infty$, $\frac{\ln u}{u} \to 0$. Hence the relevant limit of the product is 0. Note that we have used the fact that $\sin x \le x$ for all $x > 0$ with $x = \frac{1}{u}$. This result is usually proved geometrically in calculus courses for $x \in (0, \frac{\pi}{2})$ but for a more general proof one has to resort to other methods. For instance, if one knows that the infinite product representation of $\sin x = x\Big(1 - \frac{x^2}{\pi^2}\Big)\Big(1 - \frac{x^2}{2^2\pi^2}\Big)\Big(1 - \frac{x^2}{3^2\pi^2}\Big)\dots$ then it is clear that $\sin x < x$ for all $x > 0$. The following graphs say it all:





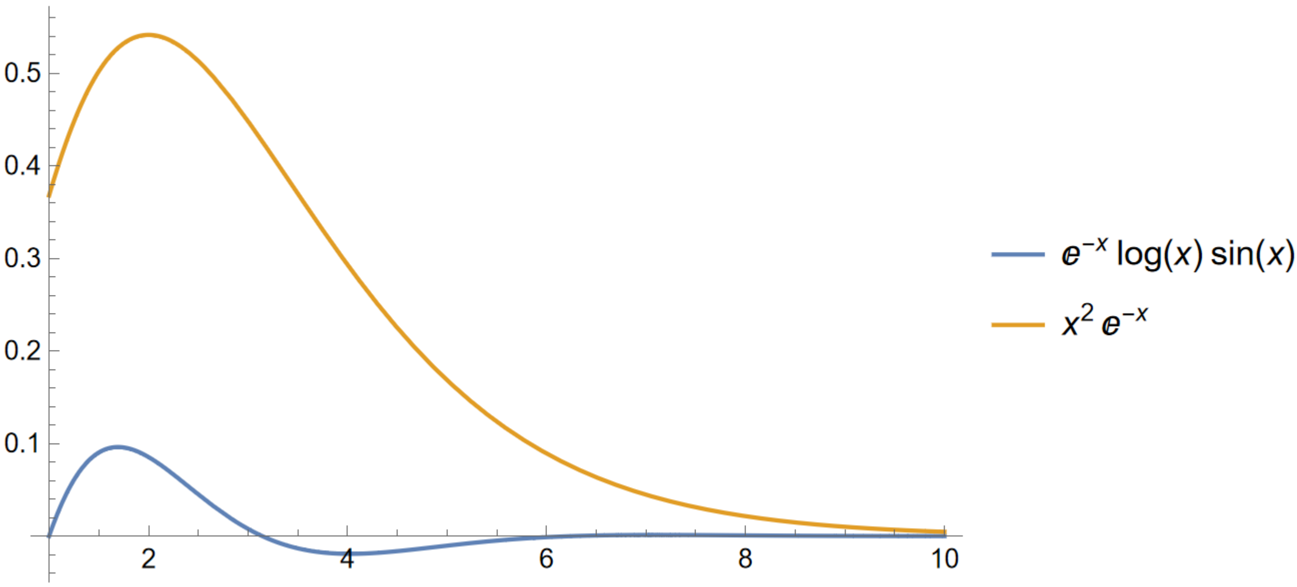
This is how $e^{-x} \ln x \sin x$ behaves as $x \to 0$.



Proving limit (2) is straightfoward using Properties 2 and 6 because we have for $x > 1$:

$$0 < e^{-x} \ln x \sin x < e^{-x} x \sin x \leq e^{-x} x^2 \to 0 \text{ by Property 2 as } x \to \infty \tag{15}$$

Here is the graphical behaviour:



2 Proofs of the Properties

Proof of Property 1

At first blush it might be thought that if $f(x)$ tends to infinity no matter how slowly we could always find some α such that x^α tends to infinity even more slowly. It turns out that the behaviour of $\ln x$ confounds this expectation.

Suppose that $\beta > 0$ is rational. Then:

$$1 < t^\beta \text{ ie } t^{-1} < t^{\beta-1} \tag{16}$$

Hence from the defintion of $\ln x$ we have:

$$\begin{aligned} \ln x =& \int_1^x \frac{dt}{t} \\ <& \int_1^x \frac{dt}{t^{1-\beta}} \\ =& \left[\frac{t^\beta}{\beta}\right]_1^x \\ =& \frac{x^\beta - 1}{\beta} \end{aligned} \tag{17}$$

Hence $\ln x < \frac{x^\beta - 1}{\beta} < \frac{x^\beta}{\beta}$ for $x > 1$. But if $\alpha > 0$ w can always choose a smaller $\beta > 0$ and then we will have:

$$0 < \frac{\ln x}{x^\alpha} < \frac{x^{\beta-\alpha}}{\beta} \tag{18}$$

But $x^{\beta-\alpha} \to 0$ when $x \to \infty$ because $\beta < \alpha$ and hence $\frac{\ln x}{x^\alpha} \to 0$ as $x \to \infty$.

The behaviour as $x \to 0^+$ can be seen from the following.

By making the substitution $x = \frac{1}{y}$ in Property 1 we have:

$$\begin{aligned} \frac{\ln x}{x^\alpha} =& \frac{\ln \frac{1}{y}}{\left(\frac{1}{y}\right)^\alpha} \\ =& - y^\alpha \ln y \end{aligned} \tag{19}$$

But as $y \to 0^+$, $x \to \infty$ and hence $\lim_{y \to 0^+} = - \lim_{x\to\infty} x^{-\alpha} \ln x = 0$.

Proof of Property 1

From Property 1 we have that $x^{-\beta}\ln x \to 0$ when $x \to \infty$ for $\beta > 0$. Let $\alpha = \frac{1}{\beta}$. Then:

$$\begin{aligned} x^{-\beta}\ln x =& (x^{-1})^{\beta}(\ln x)^{\alpha\beta} \\ =& \Big(x^{-1}(\ln x)^{\alpha}\Big)^{\beta} \to 0 \text{ as x} \to \infty \end{aligned} \tag{20}$$

This implies that $x^{-1}(\ln x)^{\alpha} \to 0$. Now let $x = e^y$. Therefore, $e^{-y}(\ln e^y)^{\alpha} \to 0$ as $y \to \infty$. In other words:

$$\frac{y^{\alpha}}{e^y} \to 0 \text{ as } y \to \infty \tag{21}$$

We can also see that if $\gamma > 0$ then $e^{\gamma y} \to \infty$ and if $\gamma < 0$ then $e^{\gamma y} \to 0$ and in each case it does this more rapidly than any power of y.

Convexity and concavity properties

It is worth noting that the exponential $e^{\alpha x}$ is convex for any $x \in \mathbb{R}$. Recall that a function $f(x)$ is convex if for all x, y in the domain of f and any θ such that $0 \le \theta \le 1$ we have that:

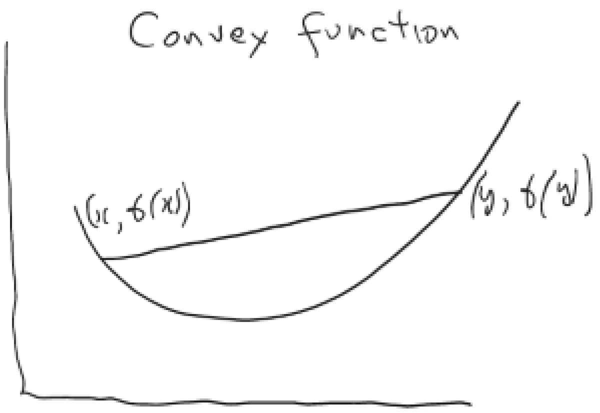
$$f(\theta x + (1-\theta)y) \le \theta f(x) + (1-\theta)f(y) \tag{22}$$

The chord between any two points on the graph lies above the graph

We say that f is concave if $-f$ is convex.

The logarithm $\ln x$ is concave for positive real numbers.

Powers of x, ie x^{α} is convex for positive reals if $\alpha \ge 1$ or $\alpha < 0$ but is concave if $0 \le \alpha \le 1$. It is useful to note in what follows that $\frac{1}{x}$ is convex on $(0, \infty)$. A pedantic point is that all linear and affine functions are convex and concave. The differential criterion for convexity is that if $f : (a, b) \to \mathbb{R}$ is twice differentiable then $f''(x) \ge 0$ for all $x \in (a, b)$ implies that f is convex on (a, b). If strict inequality holds then f is strictly convex. This is the case with $f(t) = \frac{1}{t}$ since $f''(t) = \frac{2}{t^3} > 0$ for $t > 0$.

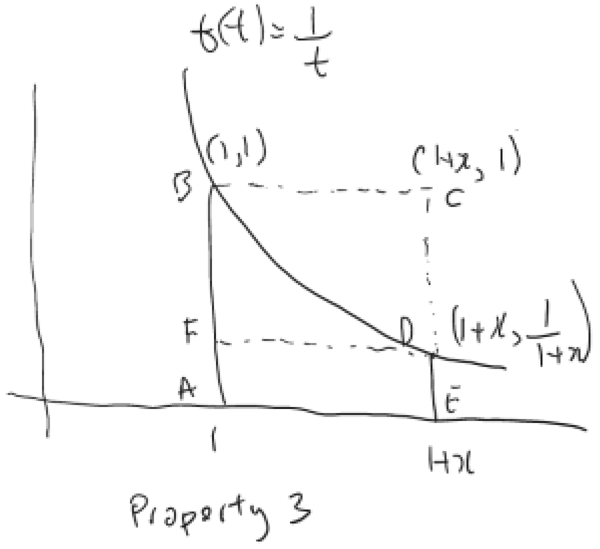


Proof of Property 3

We have to prove that:

$$\frac{x}{1+x} < \ln(1+x) < x \text{ for } x > 0 \tag{23}$$

The diagram below is the crux of solving this problem. From the definition in (12) we have that $\int_1^{1+x} \frac{dt}{t} = \ln(1+x)$ and this area is clearly less than the area of the rectangle ABCE whose area is x.1 = x. Similarly, the area at issue is greater than the area of rectangle AFDE which is $x.\frac{1}{1+x}$. Thus $\frac{x}{1+x} < \ln(1+x) < x$ for $x > 0$.

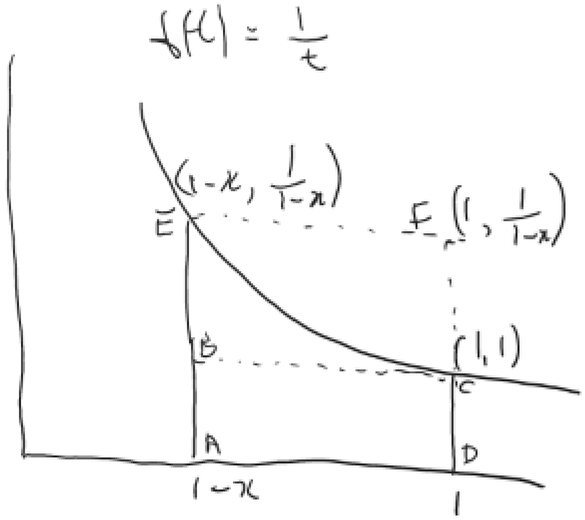


Proof of Property 4

We have to prove that:

$$x < -\ln(1-x) < \frac{x}{1-x} \text{ for } 0 < x < 1 \tag{24}$$

We have that $\int_{1-x}^{1} \frac{dt}{t} = -\ln(1-x)$ and this area is less than the area of rectangle AEFD which is $\frac{x}{1-x}$. Similarly, the area of rectangle ABCD is $x.1 = x$. Hence, given that the construction is based on $0 < x < 1$ the inequality is proved.

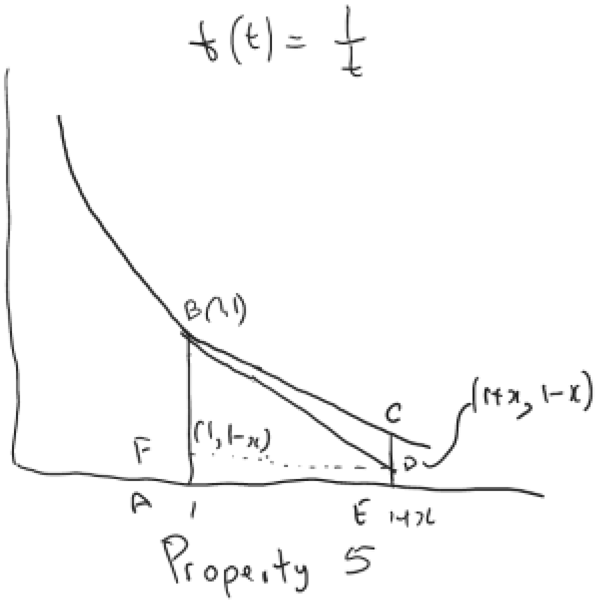


Property 4

Proof of Property 5

We have to prove that:

$$x - \frac{1}{2}x^2 < \ln(1+x) \text{ for } x > 0 \tag{25}$$



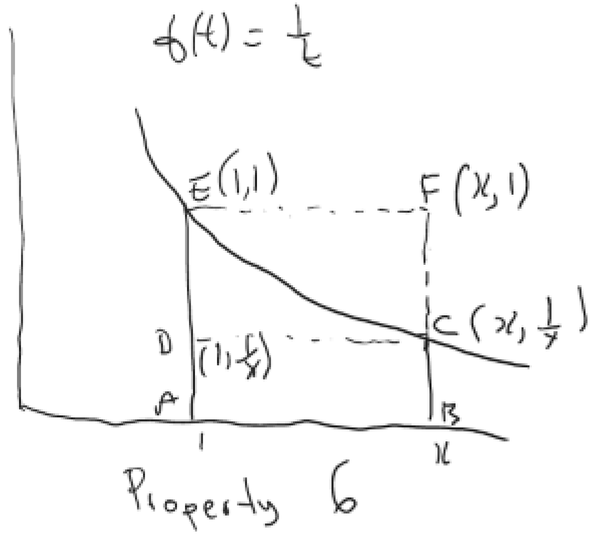
We draw a tangent at the point B(1,1) which intersects with the line $t = 1 + x$ at the point $D(1 + x, 1 - x)$. The equation of line BD is $y - 1 = -(t - 1) = 2 - t$ since the slope at of $f(t)$ is $f'(t) = -\frac{1}{t^2}$ and at (1,1) this is -1. One needs to check that this line does not intersect the curve between 1 and 1+x to ensure that the trapezoidal area AEDB is strictly less than the area under the curve. This can be done two ways. One can look to see if there is an intermediate intesection by solving $2 - t = \frac{1}{t}$. This gives rise to the quadratic $t^2 - 2t + 1 = (t - 1)^2 = 0$ which means that $t = 1$ is the only point of intersection. Alternatively one could argue about the fixed slope of the line of -1 in comparison to the slope of the curve on (1,1+x). The trapezoidal area we are after is the area of AEDB. That is:
$\ln(1 + x) = \int_1^{1+x} \frac{dt}{t} > \frac{x}{2}(1 + 1 - x) = x - \frac{x^2}{2}$.

Proof of Property 6

We have to prove that:

$$\frac{x-1}{x} < \ln x < x - 1 \text{ for } x > 1 \tag{26}$$

From the diagram, $\ln x = \int_1^x \frac{dt}{t} <$ Area rectangle ABFE $= (x-1).1 = x-1$ Similarly, $\ln x = \int_1^x \frac{dt}{t} >$ Area rectangle ABCD $= \frac{x-1}{x}$.



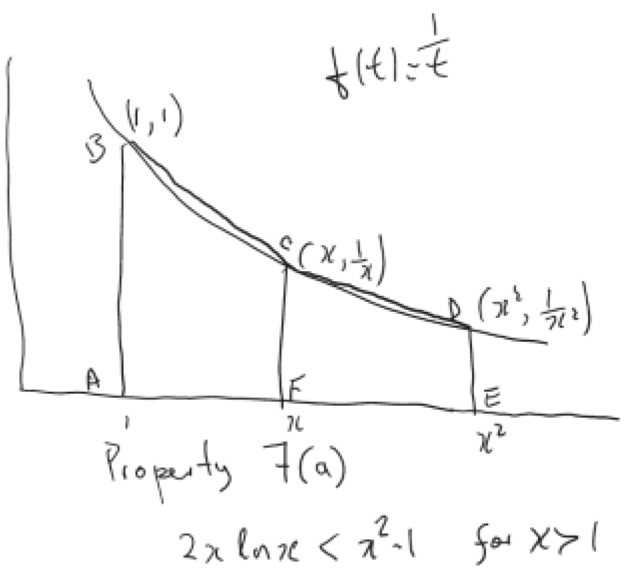
Proof of Property 7

We have to prove that:

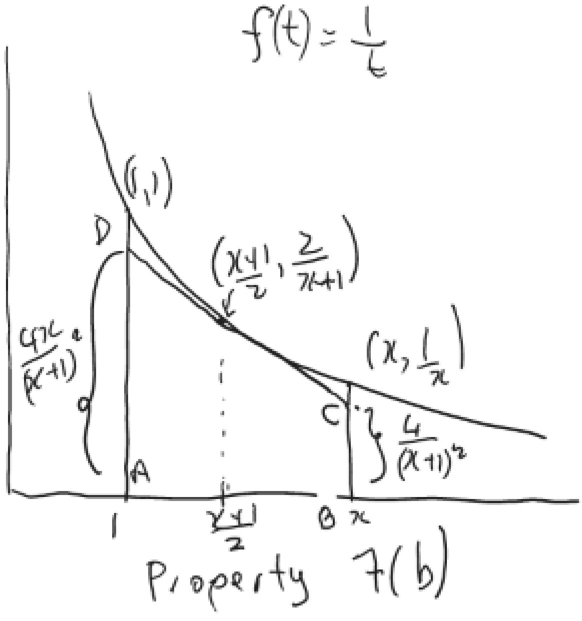
$$4(x-1) - 2\ln x < 2x\ln x < x^2 - 1 \text{ for } x > 1 \tag{27}$$

We first prove that $2x\ln x < x^2 - 1$ using the diagram below. We see that $\int_1^{x^2} \frac{dt}{t} = \ln x^2 = 2\ln x <$ Area of trapezium AFCB + Area of trapezium FEDC noting the crucial point that due to the convexity of $f(t) = \frac{1}{t}$ the chords BC and DC are both above the curve so that the area under the curve is indeed dominated by the sum of these two trapeziums. Thus we have that:

$$
\begin{aligned}
2\ln x <& \frac{(x-1)}{2}(1+\frac{1}{x}) + \frac{x^2-x}{2}(\frac{1}{x}+\frac{1}{x^2}) \\
=& \frac{x^2-1}{2x} + \frac{(x^2-x)}{2}\frac{(x^2+x)}{x^3} \\
=& \frac{x^2(x^2-1)+x^2(x^2-1)}{2x^3} \\
=& \frac{x^2-1}{x} \\
\Longrightarrow & 2x\ln x < x^2-1
\end{aligned}
\tag{28}
$$



To prove the other side of the inequality we need another diagram:



We see that the area of the trapezium ABCD is less than $\int_1^x \frac{dt}{t} = \ln x$. We construct the straight line segment tangent to the curve at $(\frac{x+1}{2}, \frac{2}{x+1})$, the end points of which are easily shown to lie below the points $(1, 1)$ and $(x, \frac{1}{x})$. The equation of the required line is:

$$y - \frac{2}{x+1} = -\frac{4}{(x+1)^2}(t - \frac{(x+1)}{2}) \\ \therefore y = -\frac{4t}{(x+1)^2} + \frac{x}{x+1} \tag{29}$$

When $t = x$ the y-intercept is:

$$y = \frac{-4x + 4x + 4}{(x+1)^2} = \frac{4}{(x+1)^2} \tag{30}$$

Note that $\frac{4}{(x+1)^2} < \frac{1}{x}$ because $(x-1)^2 > 0$ since $x > 1$.
Similarly when $t = 1$ the y-intercept is:

$$y = \frac{-4+4x+4}{(x+1)^2} = \frac{4x}{(x+1)^2} \tag{31}$$

Note that $\frac{4x}{(x+1)^2} < 1$ again because $(x-1)^2 > 0$ since $x > 1$.

Thus we have that:

$$\begin{aligned}\int_1^x \frac{dt}{t} =& \ln x \\ &> \text{Area of trapezium ABCD} \\ &= \Big(\frac{x-1}{2}\Big)\Big(\frac{4x}{(x+1)^2} + \frac{4}{(x+1)^2}\Big) \\ &= 4\frac{(x-1)}{2}\frac{(x+1)}{(x+1)^2} \\ &= 2\frac{(x-1)}{x+1}\end{aligned} \tag{32}$$

Thus we have shown that:

$\ln x > 2\frac{(x-1)}{x+1}$ but this implies that $4(x-1) - 2\ln x < 2x\ln x$ which as required to be proved.

Proof of Property 8

We have to prove that:

$$0 < \frac{1}{x} - \ln\frac{x+1}{x} < \frac{1}{2x^2} \text{ for } x > 0 \tag{33}$$

Putting $x = \frac{1}{u}$ where $u > 0$ into Property 5 we have:

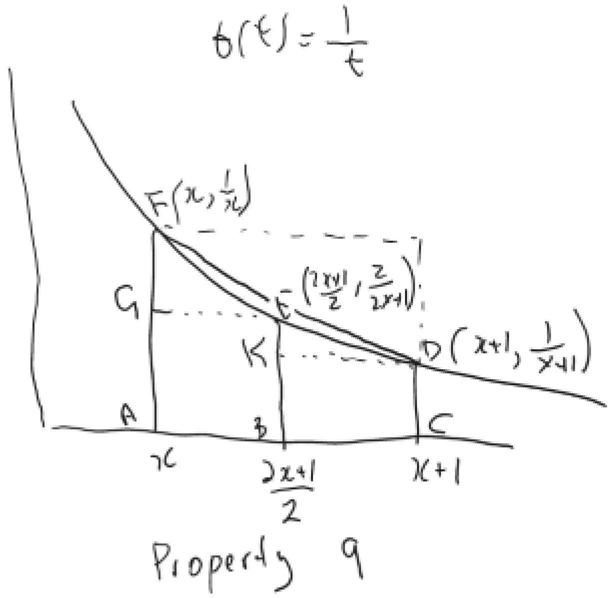
$$\begin{aligned}\frac{1}{u} - \frac{1}{2u^2} <& \ln(1+\frac{1}{u}) \\ 0 <& \frac{1}{u} - \ln\frac{1+u}{u} < \frac{1}{2u^2}\end{aligned} \tag{34}$$

Proof of Property 9

We have to prove that:

$$\frac{2}{2x+1} < \ln\frac{x+1}{x} < \frac{2x+1}{2x(x+1)} \text{ for } x > 0 \tag{35}$$

To prove this inequality we use the following diagram:



We have that for $x > 0$, $\ln\frac{1+x}{x} = \int_x^{1+x}\frac{dt}{t} >$ Area of trapezium CDFA $= \frac{1}{2}(\frac{1}{1+x} + \frac{1}{x}) = \frac{2x+1}{2x(x+1)}$

Similarly, $\ln\frac{1+x}{x} = \int_x^{1+x}\frac{dt}{t} >$ Area of rectangle ABEG + Area rectangle CDKB. Therefore:

$$\begin{aligned}\ln\frac{1+x}{x} >& \Big(\frac{2x+1}{2} - x\Big) \times \frac{2}{2x+1} + \Big(x+1-(\frac{2x+1}{2})\Big)\frac{1}{x+1}\\ =&\frac{1}{2x+1} + \frac{1}{2x+1}\\ =&\frac{2}{2x+1}\end{aligned} \tag{36}$$

Proof of Property 10:

We have to prove that:

$$\lim_{x\to 1}\frac{\ln x}{x-1}=\lim_{y\to 0}\frac{\ln(1+y)}{y}=1 \tag{37}$$

From Properrty 6 we have that for $x>1$:

$\frac{1}{x}<\frac{\ln x}{x-1}<1$ and hence as $x\to 1^+, \frac{1}{x}\to 1$ and by the "Sandwich Principle" $\lim_{x\to 1}\frac{\ln x}{x-1}=1$.

Using Property 3 we see that for $y>0$:

$$\begin{aligned}\frac{y}{1+y}<\ln(1+y)<y\\ \frac{1}{1+y}<\frac{\ln(1+y)}{y}<1\end{aligned} \tag{38}$$

Therefore by the "Sandwich Principle" as $y\to 0^+$, $\frac{\ln(1+y)}{y}\to 1$.

[1] G H Hardy, "A Course in Pure Mathematics", Tenth Edition, Cambridge University Press, 2006

3 History

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