Is there a rigorous high school limit proof that $0^0 = 1$?

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1 A bare hands proof

Youtube contains a number of videos seemingly pitched at high school students in which attempts are made to explain why the $\lim_{x\to 0^+} x^x = 1$. Some of the presenters are high school teachers and one in particular is a local Australian one, Eddie Woo, whose video on the subject has had about 1,273,000 views with 25,000 likes: `https://www.youtube.com/watch?v=r0_mi8ngNnM&t=2s`

What he does is get the students to calculate x^x on their calculators with values in the interval (0,1). As x decreases from 1 to close to 0 the value of x^x decreases then increases. When asked by a student about this behaviour he gives no explanation which astonished me. In short his video is little more than a very basic bit of numerical exploration and certainly gives no real understanding of what is going on. Other videos wheel out L'Hopital's rule but this is hardly appropriate for high school students any more than a sequential approach would be since the students lack the basic infrastructure of a first course in real analysis.

This particular problem is usually done in a first year university analysis course or perhaps calculus course and is usually put to rest with L'Hopital's rule although there are "hard core" sequential style approaches. However, is it possible to provide a semi-rigorous argument using only some basic calculus available to a high school student? I think the answer is "yes" but I also think all but the most keen students would find it too sophisticated given the Youtube reactions to the essentially low-beer explanations that are popular simply because of a cheesy delivery pitched at a low level.

Let's start with some basic points.

We know that $a^n = \underbrace{a \times a \times a \cdots \times a}_{\text{n times}}$ and that $\frac{a^n}{a^n} = 1 = a^{n-n} = a^0$ for any $a \neq 0$. We might think that 0^0 is zero on that basis – 0 being multiplied 0 times. That is not an entirely unreasonable view. However, if we write x^x in terms of the exponential a new way of looking at the problem opens up.

Students should be aware that:

$$x^x = e^{x \ln x} \tag{1}$$

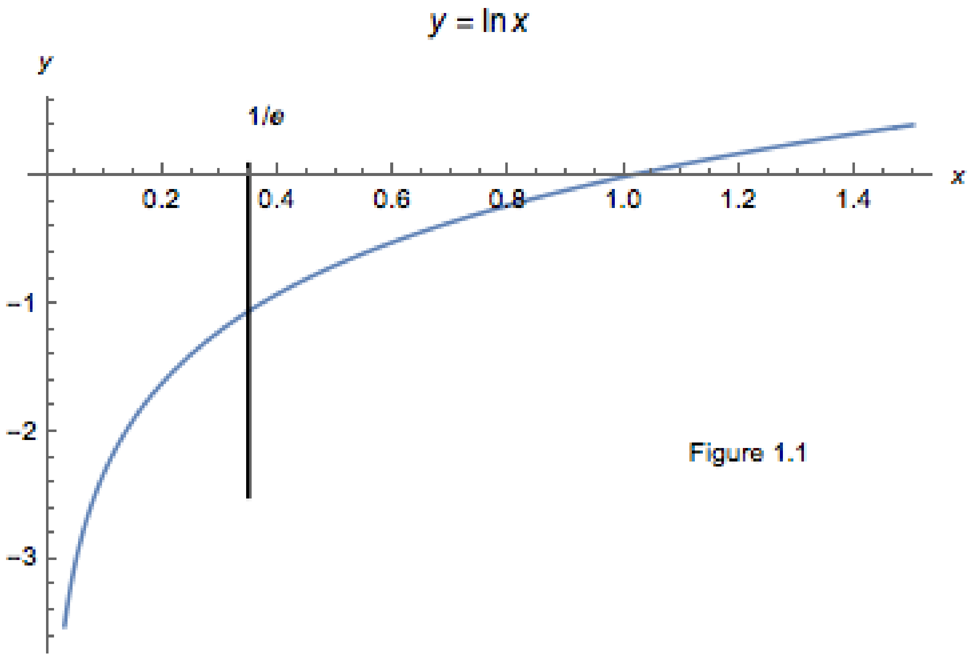
The proof is simply one of definition:

Let $x^x = y$ then $\ln x^x = x\ln x = \ln y$ from which it follows that $x^x = e^{\ln y} = e^{x\ln x}$

Because we have now recast x^x in terms of the exponential function we immediately get all the wonderful properties it possesses – it is continuous everywhere (actually uniformly continuous on any closed interval), infinitely differentiable and has a convergent powers series. In short you can do the mathematical equivalent of terrible things to small furry animals to the exponential function without getting arrested! To the extent that we will need some "faith based" maths we can be sure that no student will be misled. I have looked at a few videos and am astonished that they do not refer to the wondrous properties of the exponential function. Not only is this disrespectful to Euler's profound genius, it is also redolent of deep mathematical ignorance.

We can use basic calculus to get a good idea of what is going on. The first derivative tells us a lot.

If $f(x) = e^{x\ln x}$ then $f^{'}(x) = (1+\ln x)e^{x\ln x}$ and we see that it is zero when $1+\ln x = 0$ ie $\ln x = -1$ which occurs when $x = \frac{1}{e}$. The second derivative is : $f''(x) = \left((1+\ln x)^2 + \frac{1}{x} \right) e^{x\ln x}$ and this is positive for all $x \in (0,1)$. This means there really is a minimum at $x = \frac{1}{e}$. For $x \in (\frac{1}{e}, 1)$, $f'(x) > 0$ and for $x \in (0, \frac{1}{e})$, $f^{'}(x) < 0$. See Figure 1.1 In both cases the gradient is large due to the factor $e^{x\ln x}$ but because $|\ln x| > |\ln \frac{1}{e}|$ for $0 < x < \frac{1}{e}$ and $|\ln x| < |\ln \frac{1}{e}|$ for $\frac{1}{e} < x < 1$, the graph of $e^{x\ln x}$ must be steeper on $(0, \frac{1}{e})$ than on $(\frac{1}{e}, 1)$. See Figure 1. For $x > 1$ the graph is simply exponential in character.



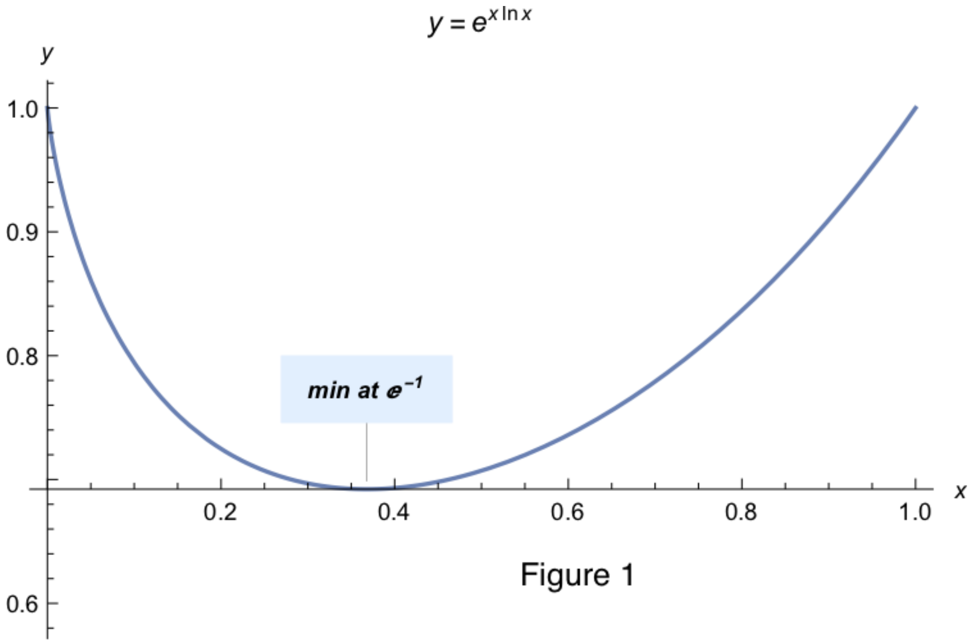


Figure 1

Note that the behaviour of $f(x) = e^{x \ln x}$ is monotonic in each of the subintervals a $(0, \frac{1}{e})$ and $(\frac{1}{e}, 1)$. This can be proved as follows. For $0 < x' < x < \frac{1}{e}$ we have $\ln x' < \ln x < 0$. Therefore, bearing in mind $0 < x' < x$, $-x' > -x$ and $|\ln x'| > |\ln x| > 0$ we have:

$$x' \ln x' = -x' |\ln x'| > -x' |\ln x| > -x |\ln x| = x \ln x \tag{2}$$

Hence it follows that $f(x') > f(x)$. The same logic applies to the other subinterval but is it not important in this argument.

Looking at Figure 1 it appears suspiciously quadratic in character (at least to the resolution of the eye) and that is because it is. This is where some faith based mathematics comes in.

We can give a power series (Taylor's series) expansion for $e^{x \ln x}$ as follows:

$$e^{x \ln x} = 1 + x \ln x + \frac{(x \ln x)^2}{2!} + \text{remainder} \tag{3}$$

If we can ignore the remainder term – and we can for small x – we see that we can approximate $e^{x \ln x}$ by something quadratic in $x \ln x$ ie $1 + x \ln x + \frac{(x \ln x)^2}{2!}$. As we approach zero from 1, the function decreases monotonically to the minimum and then increases monotonically and looks like it is getting close to 1. Could the function exceed 1? Let's consider a series of values $x_n = \frac{1}{n}$ for $n \geq 1$. Clearly $x_n > 0$ for all n and we can get as close to 0 as we like by making n arbitrarily large.

Thus we have:

$$e^{x_n \ln x_n} = e^{\frac{1}{n} \ln \frac{1}{n}} = e^{\frac{-\ln n}{n}} \tag{4}$$

We actually know a lot about how $e^{\frac{-\ln n}{n}}$ behaves as n gets bigger and bigger. In Figure 2 there is a graph of $y = \ln n$ and the line $y = n$. Clearly, if a picture tells a thousand words it tells us that for $n > 1$, $n > \ln n$ so that $0 < \frac{\ln n}{n} < 1$. But we also know that for $u > 0$, $e^{-u} < 1$ (see Figure 3). Here $u = \frac{\ln n}{n} > 0$ so we have a contradiction since we supposed that $e^{x \ln x} > 1$. Hence, $e^{x_n \ln x_n} \leq 1$ and in the limit equals 1.

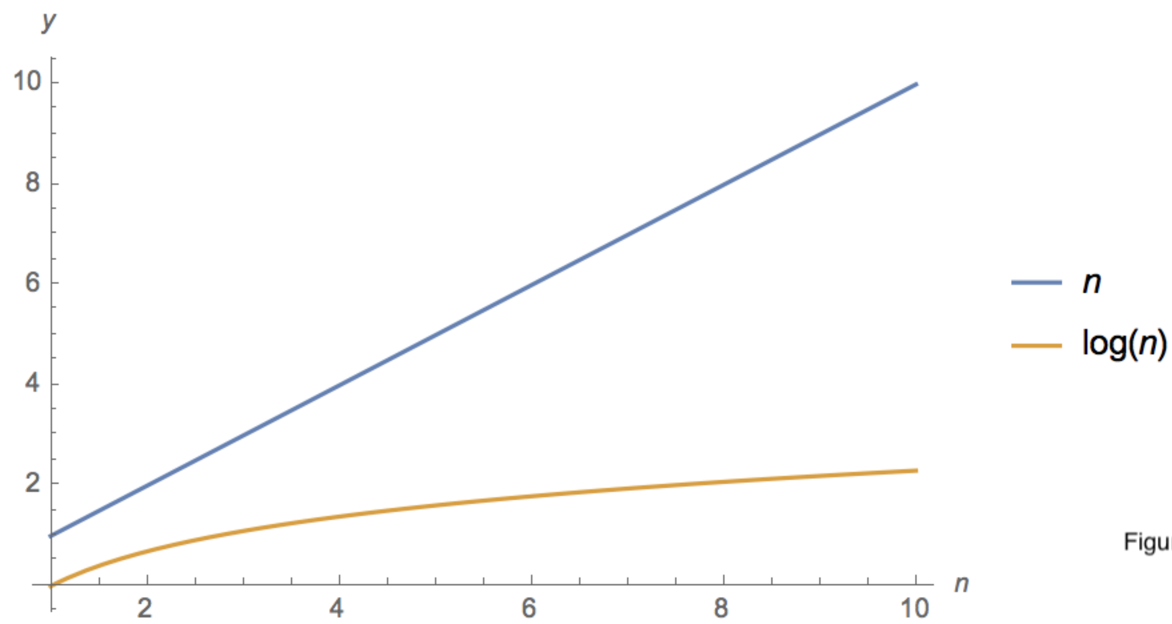


Figure 2

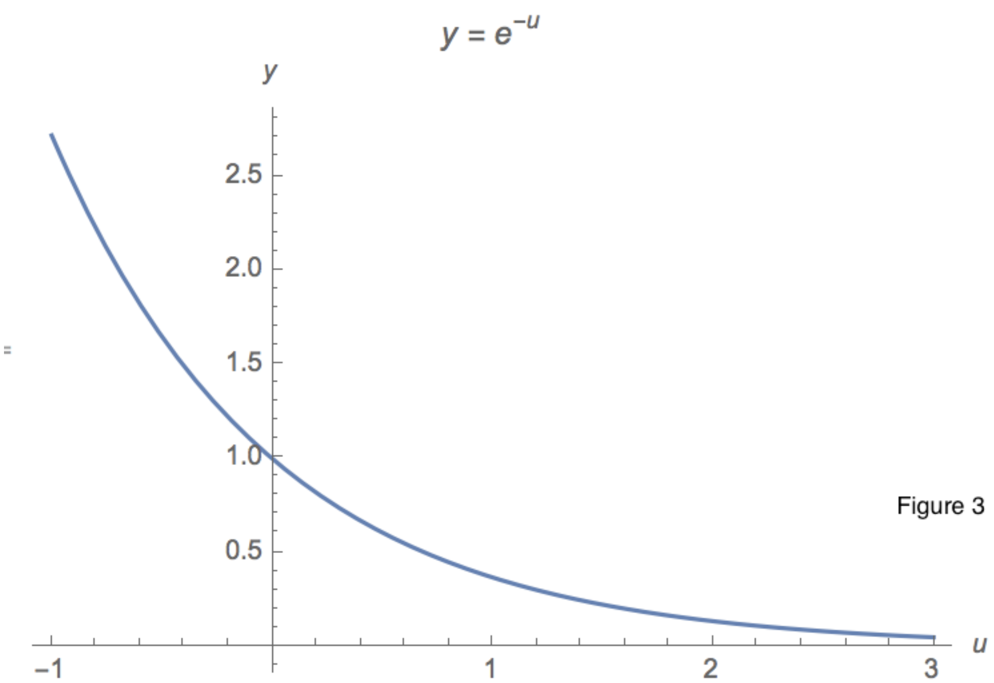


Figure 3

If we feel bolder we can even attack the problem directly with the definition of limit which is not developed rigorously in most high school courses. What the definition says is this. You give me any positive ϵ , no matter how small, and I'll produce a positive δ so that $|e^{x \ln x} - 1| < \epsilon$ whenever $0 < x < \delta$. This is what $\lim_{x\to 0^+} e^{x \ln x} = 1$ means. I am definitely not trying to explain the whole intellectual edifice of limits here by the way. If you make ϵ smaller you necessarily squeeze δ. Because we are only concerned with the behaviour of $e^{x \ln x}$ very close to 0, let's focus on the interval $(0, \frac{1}{e})$ because we already know quite a bit about the properties of the function there. Now if $0 < x < \delta < \frac{1}{e}$ we have that:

$$|e^{x \ln x} - 1| = 1 - e^{x \ln x} \tag{5}$$

Why? Recall that $|x| = x$ if $x > 0$ and $-x$ if $x < 0$. We know that for $x \in (0, \frac{1}{e})$, $e^{x \ln x} \leq 1$ so $|e^{x \ln x} - 1| = -(e^{x \ln x} - 1) = 1 - e^{x \ln x}$.

But we also know that for $0 < x < \delta < \frac{1}{e}$:

$$\begin{aligned} e^{x \ln x} &> e^{\delta \ln \delta} \\ \therefore -e^{x \ln x} &< -e^{\delta \ln \delta} \\ \therefore 1 - e^{x \ln x} &< 1 - e^{\delta \ln \delta} \end{aligned} \tag{6}$$

So we need to find a δ such that $1 - e^{\delta \ln \delta} < \epsilon$ ie:

$$1 - \epsilon < e^{\delta \ln \delta} \tag{7}$$

How do we find the δ that satisfies (7) ? The easiest way is an approximation method that uses the properties we have already established for $e^{x \ln x}$. Equation (7) is a transcendental equation and hence you will struggle to get a closed form answer!!

When $\delta = 0.001$, $e^{\delta \ln \delta} = 0.993116$
When $\delta = 0.0001$, $e^{\delta \ln \delta} = 0.999079$

Thus if we take $0 < x < 0.0001$ we can be assured that $1 - e^{x \ln x} < 0.001$. Doubt me? Try $x = 0.00005$ say. Then $1 - e^{x \ln x} = 1 - 0.999505 = 0.000495 < 0.001$ (see (5)).

2 Proof using L'Hopital's rule

The $\frac{\infty}{\infty}$ form of L'Hopital's rule provides a quick proof of the limit we are after. Of course, its use presumes a knowledge of why it works (and it does in this case) and the full proofs of its validity are not usually given at high school level. We simply rewrite the exponent $x \ln x = \frac{\ln x}{\frac{1}{x}} = \frac{g(x)}{h(x)}$ and then we see that as $x \to 0^+$, the exponent takes the form of $-\frac{\infty}{\infty}$. We can forget about the minus sign for reasons which will become clear. L'Hopital's rule tells us that $\lim_{x\to 0^+} \frac{g(x)}{h(x)} = \lim_{x\to 0^+} \frac{g'(x)}{h'(x)}$. Thus

$$\lim_{x\to 0^+} e^{\frac{g(x)}{h(x)}} = e^{\lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-1}{x^2}}} = e^{\lim_{x\to 0^+} -x} = e^0 = 1 \tag{8}$$

Implicitly a continuity property has been used here ie the assertion that $\lim_{x\to 0^+} e^{p(x)} = e^{\lim_{x\to 0^+} p(x)}$ and that is completely kosher but I am not going to prove it.

3 But L'Hopital's rule is for the slob mathematician !

L'Hopital's rule works but in this case I think we can convince a high school student with basic calculus that the following limit exists:

$$\lim_{x\to 0^+} x \ln x = 0 \tag{9}$$

Assuming the student buys that argument and if they are prepared to take the continuity argument referred to above on faith (and after all it is the exponential function with all its wondrous properties) you get to the same result as you do using L'Hopital's rule.

So let $\beta > 0$ be some rational number. By definition:

$$\ln y = \int_1^y \frac{dt}{t} \tag{10}$$

This inequality holds (because $t > 1$ and $\beta > 0$):

$$t^{-1} < t^{\beta} t^{-1} = t^{\beta - 1} \tag{11}$$

Therefore:

$$\ln y = \int_1^y \frac{dt}{t} < \int_1^y \frac{dt}{t^{1-\beta}} = \frac{y^{\beta} - 1}{\beta} < \frac{y^{\beta}}{\beta} \text{ which holds for } y > 1 \tag{12}$$

If $\alpha > 0$ we can always choose a smaller positive β and then we have:

$$0 < \frac{\ln y}{y^{\alpha}} < \frac{y^{\beta}}{\beta y^{\alpha}} = \frac{y^{\beta-\alpha}}{\beta} \tag{13}$$

But $y^{\beta-\alpha} \to 0$ as $y \to \infty$ since $\beta < \alpha$. Therefore:

$$y^{-\alpha} \ln y \to 0 \text{ as } y \to \infty \tag{14}$$

That this is the case follows from (13) because we have "sandwiched" $\frac{\ln y}{y^{\alpha}}$ with something which approaches 0 (ie $\frac{y^{\beta-\alpha}}{\beta}$) from the top with 0 on the bottom.

Now let $x = \frac{1}{y}$ in (14) where $y \to \infty$ so that $x \to 0^+$.

So :

$$\left(\frac{1}{x}\right)^{-\alpha} \ln\left(\frac{1}{x}\right) = -x^{\alpha} \ln x \to 0 \text{ as } x \to 0^+ \tag{15}$$

ie $x^{\alpha} \ln x \to 0$ as $x \to 0^+$. So put $\alpha = 1$ and we have $\lim_{x\to 0^+} x \ln x = 0$. Finally using the continuity faith based assumption we have that $\lim_{x\to 0^+} e^{x \ln x} = e^{\lim_{x\to 0^+} x \ln x} = e^0 = 1$.

4 Conclusions

Now I am not naive enough to believe that your average or even mildly interested student is going to get excited by the sort of analysis set out above. Analysis courses at university level are usually shunned by all but the most committed students because of the high standard of reasoning needed. However, there is nothing in section 1 that a final year high school student could not do.

5 History

Created

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