Laplace's method for integral asymptotics

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September 28, 2015

1 Introduction

Laplace's method is an extension of the theory behind Laplace transforms which enables one to extract the leading order behaviour of certain integrals. The theory of Laplace transforms enables us to quickly evaluate an integral such as (1) without actually performing an integration:

$$\int_0^\infty t\, e^{-5t}\, dt \tag{1}$$

Recalling how this is done we know that the Laplace transform of a function $f(t)$ is defined for $s > 0$ as:

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)\, e^{-st}\, dt \tag{2}$$

The transform will exist if f is piecewise continuous on $0 < t < A$ for any $A > 0$ and $f(t)$ is of exponential order as $t \to \infty$ ie there are positive constants C, M and a constant a such that $|f(t)| \le Ce^{at}$ when $t \ge M$. Under these conditions the Laplace transform will exist for $s > a$ because of the following estimates:

$$\begin{aligned} \Big|\mathcal{L}\{f(t)\}\Big| &= \Big|\int_0^M f(t)\, e^{-st}\, dt + \int_M^\infty f(t)\, e^{-st}\, dt\Big| \\ \le & \underbrace{\Big|\int_0^M f(t)\, e^{-st}\, dt\Big|}_{\text{the continuity of f assumption ensures this is finite}} + \Big|\int_M^\infty f(t)\, e^{-st}\, dt\Big| \\ &\le \text{constant} + \int_M^\infty Ce^{(a-s)t}\, dt \end{aligned} \tag{3}$$

Thus if $a < s$ the final integral in (3) will exist and hence the Laplace transform will exist. To solve our banal integral in (1) we simply need to find the Laplace transfrom of the following:

$$\mathcal{L}\{t\} = \int_0^\infty t\, e^{-st}\, dt = \underbrace{\Big[\frac{-te^{-st}}{s}\Big]_0^\infty}_{=0} + \frac{1}{s}\int_0^\infty e^{-st}\, dt = \frac{1}{s^2} \tag{4}$$

Hence the integral we want has $s = 5$ and so $\int_0^\infty t\, e^{-5t}\, dt = \frac{1}{25}$.

A more significant problem is this: Let n be an integer with $n \to \infty$. Using the fact that:
$\int_0^{\frac{\pi}{2}} \sin^{2n} x\, dx = \int_0^{\frac{\pi}{2}} \cos^{2n} x\, dx = \frac{1.3....(2n-1)}{2.4....2n}\frac{\pi}{2}$ prove that:

$$\frac{1.3.\dots(2n-1)}{2.4.\dots 2n} \sim \frac{1}{\sqrt{n\pi}} \tag{5}$$

The symbol $\sim$ means that the LHS is asymptotically equal to the RHS. To say that $a_n \sim b_n$ means that $b_n \neq 0$ for sufficiently large n and $\frac{a_n}{b_n} \to 1$ as $n \to \infty$. Note that (5) is effectively Wallis' formula ie

$$\lim_{n\to\infty} \frac{2.2.4.4.6.6.\dots(2n).(2n)}{1.3.3.5.5.\dots(2n-1).(2n-1).(2n+1)} = \frac{\pi}{2} \tag{6}$$

To see this, invert (5), square, multiply by $\frac{1}{2n+1}$ and let $n \to \infty$.

Note that one can prove $\int_0^{\frac{\pi}{2}} \sin^{2n} x\, dx = \int_0^{\frac{\pi}{2}} \cos^{2n} x\, dx = \frac{1.3....(2n-1)}{2.4....2n}\frac{\pi}{2}$ by integral reduction methods and induction. As will be shown below, the Laplace method obviates all of those steps.

This problem comes from the famous book by George Polya and Gabor Szegö [[2] ,Problem 202, page 97] and is used to demonstrate the Laplace method of asymptotic evaluation of integrals. But first one has to prove why the Laplace method works and this too is a problem in the book [[2] Problem 201, pages 96-97].

2 Statement of Laplace's method and proof

Polya and Szegö set up the hypotheses of the Laplace method as follows:

The functions $\phi(x)$, $h(x)$ and $f(x) = e^{h(x)}$ are defined on the finite or infinite interval $[a, b]$ and satisfy the following conditions:

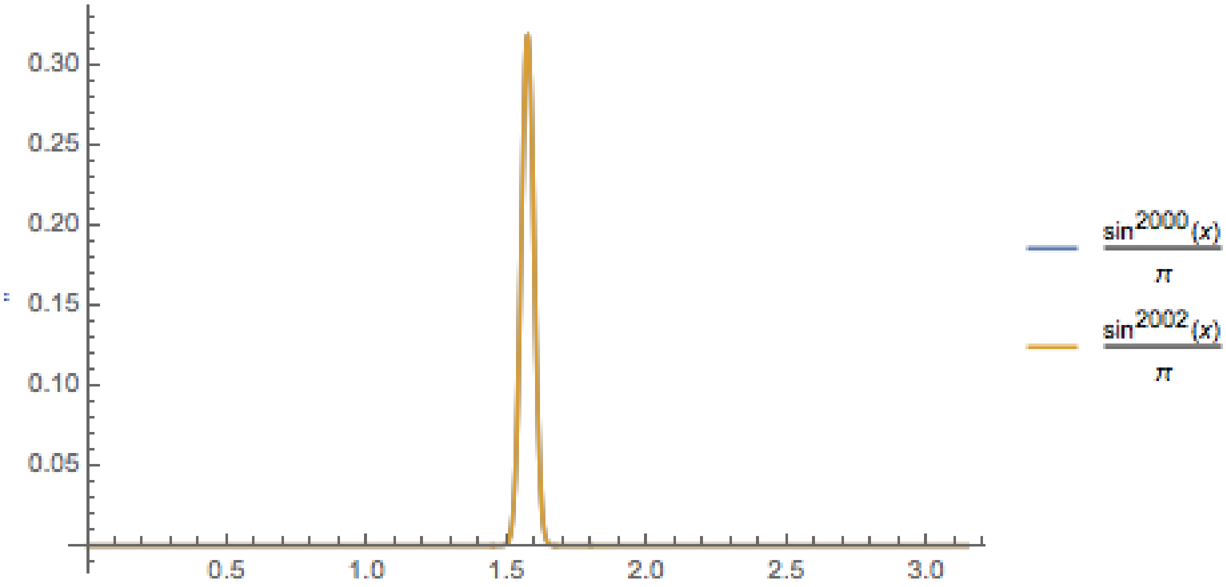
1. $\phi(x)\,[f(x)]^n = \phi(x)\,e^{n\,h(x)}$ is absolutely integrable over $[a, b]$ for $n = 0, 1, 2, \ldots;$
2. The function $h(x)$ attains it maximum only at the point ξ in (a, b); moreover, the least upper bound of $h(x)$ is smaller than $h(\xi)$ on any closed interval that does not contain ξ; there is, furthermore, a neigbourhood of ξ where $h''(x)$ exists and is continuous; finally $h''(\xi) < 0$;
3. $\phi(x)$ is continuous at $x = \xi$, $\phi(\xi) \neq 0$.

Then the following asymptotic formula holds as $n \to \infty$:

$$\int_a^b \phi(x)\,[f(x)]^n\,dx \sim \phi(\xi)\,[f(\xi)]^{n+\frac{1}{2}} \sqrt{-\frac{2\pi}{n\,f''(\xi)}} = \phi(\xi)\,e^{n\,h(\xi)} \sqrt{-\frac{2\pi}{n\,h''(\xi)}} \qquad (7)$$

The authors give the following hint for a solution: We only consider a neighbourhood of ξ and expand $h(x)$ in powers of $(x - \xi)$ up to terms of second order.

Before launching into any hard core analysis, let's look at some qualitative aspects of the problem. In the graph below the function $\frac{\sin^{2n} x}{\pi}$ is graphed for $n = 1000, 1001$ on $[0, \pi]$. The value of the integral of this function is dominated by the behaviour at $x = \frac{\pi}{2}$ which is where the maximum of the function occurs.



Thus in terms of estimating $\int_0^{\pi} \frac{\sin^{2n} x}{\pi}\,dx$ (the reason for the $\frac{1}{\pi}$ and interval $[0, \pi]$ will become clearer shortly) we would like to be able to say something like this:

$$\int_0^{\pi} \frac{\sin^{2n} x}{\pi}\, dx = \underbrace{\int_0^{\frac{\pi}{2}-\epsilon} \frac{\sin^{2n} x}{\pi}\, dx}_{\text{dominated by middle term}} + \int_{\frac{\pi}{2}-\epsilon}^{\frac{\pi}{2}+\epsilon} \frac{\sin^{2n} x}{\pi}\, dx + \underbrace{\int_{\frac{\pi}{2}+\epsilon}^{\pi} \frac{\sin^{2n} x}{\pi}\, dx}_{\text{dominated by middle term}} \sim \int_{\frac{\pi}{2}-\epsilon}^{\frac{\pi}{2}+\epsilon} \frac{\sin^{2n} x}{\pi}\, dx \tag{8}$$

In other words, the two tails of the function give rise to a relatively negligible contribution. Thus the hypotheses of the Laplace method are designed to ensure that one can confidently conclude that the leading term behaviour of the integral as n gets bigger is encoded in the behaviour of the integral around the maximum of the integrand.

The proof given by Polya and Szeg*ö* runs like this and is quite brief so I will expand on the details.

Let $\epsilon > 0$, δ be positive and so small that $a < \xi - \delta < \xi + \delta < b$ and:

(i) $\phi(\xi) - \epsilon < \phi(x) < \phi(\xi) + \epsilon$;

(ii) $h''(\xi) - \epsilon < h''(x) < h''(\xi) + \epsilon < 0$ whenever $\xi - \delta < x < \xi + \delta$.

Then:

$$\begin{aligned} \int_a^b \phi(x)\, e^{n[h(x)-h(\xi)]}\, dx &= \int_{\xi-\delta}^{\xi+\delta} \phi(x)\, e^{n[h(x)-h(\xi)]}\, dx + O(\alpha^n) \\ &= \phi(\xi') \int_{\xi-\delta}^{\xi+\delta} e^{\frac{n(x-\xi)^2 h''(\xi'')}{2}}\, dx + O(\alpha^n) \end{aligned} \tag{9}$$

where $0 < \alpha < 1$ and α depends on ϵ but not on n, $\xi - \delta < \xi' < \xi + \delta$, and $\xi - \delta < \xi'' < \xi + \delta$. Note that $O(a_n)$ with $a_n > 0$ means that there is some constant $C > 0$ such that $\frac{O(a_n)}{a_n} \le C$. The first term on the last line of the RHS of (9) lies between:

$$[\phi(\xi) - \epsilon] \int_{\xi-\delta}^{\xi+\delta} \phi(x)\, e^{\frac{n\,[(x-\xi)^2 h''(\xi'')-\epsilon]}{2}}\, dx \text{ and } [\phi(\xi) + \epsilon] \int_{\xi-\delta}^{\xi+\delta} \phi(x)\, e^{\frac{n\,[(x-\xi)^2 h''(\xi'')+\epsilon]}{2}}\, dx \tag{10}$$

The authors finally note that the bounds in (10) are asymptotically equal to:

$$[\phi(\xi) - \epsilon] \sqrt{-\frac{2\pi}{n[h''(\xi) - \epsilon]}} \text{ and } [\phi(\xi) + \epsilon] \sqrt{-\frac{2\pi}{n[h''(\xi) + \epsilon]}} \tag{11}$$

Thus since $\epsilon > 0$ is arbitrary we get the result that:

$$\int_a^b \phi(x)\,[f(x)]^n\,dx \sim \phi(\xi)\,e^{n\,h(\xi)}\,\sqrt{-\frac{2\pi}{n\,h''(\xi)}} \tag{12}$$

Note here that :

$$\int_a^b \phi(x)\,e^{n\,h(x)}\,dx = e^{n\,h(\xi)}\int_a^b \phi(x)\,e^{n[h(x)-h(\xi)]}\,dx \tag{13}$$

The step in (9) is the crux of the proof since it reflects the fact that the leading term of the integral arises from the a neighbourhood around the maximum of $h(x)$ at $x = \xi$. Because $\alpha < 1$ the other term will be negligible for large n. In fact this term encodes the size the two tails. To prove this we have to get an estimate for the tails:

$$T_1 = \int_a^{\xi-\delta} \phi(x)\,e^{n[h(x)-h(\xi)]}\,dx \quad \text{and } T_2 = \int_{\xi+\delta}^b \phi(x)\,e^{n[h(x)-h(\xi)]}\,dx \tag{14}$$

Since the logic is the same for both tails we only need to focus on T_1. By assumption we know that $\sup_x\{h(x)\} < h(\xi)$ where x is in any closed interval that does not contain ξ. This means that $h(x) - h(\xi) < 0$ on all such intervals - in particular $[a, \xi-\delta]$. This in turn means that $e^{n[h(x)-h(\xi)]} < 1$. So let $\alpha = \sup_x\{h(x)\}$ and then $0 < \alpha < 1$. This is critical because if $\alpha \geq 1$ the tail terms will not be small as $n \to \infty$. Note that α implicitly depends on ϵ because of the way δ is defined.

Thus:

$$|T_1| \leq \int_a^{\xi-\delta} |\,\phi(x)\,e^{n[h(x)-h(\xi)]}|\,dx \leq \int_a^{\xi-\delta} |\,\phi(x)\,|\,\alpha^n\,dx \leq B\alpha^n \tag{15}$$

for some $B > 0$. This follows from the assumption that $\phi(x)\,[f(x)]^n = \phi(x)\,e^{n\,h(x)}$ is absolutely integrable on $[a, b]$ for $n = 0, 1, 2\ldots.$ Thus the integral $\int_a^{\xi-\delta}|\,\phi(x)\,|\,dx$ must exist as a finite number and so will bounded by B . Note that it is not assumed that $\phi(x)$ is continuous on $[a, b]$. If that were the case then clearly $\phi(x)$ is bounded on $[a, b]$. Thus the sum of the two tails is $O(\alpha^n)$ which becomes negligible as $n \to \infty$.

In (9) the authors are also basically asserting that:

$$\int_{\xi-\delta}^{\xi+\delta} \phi(x)\,e^{n[h(x)-h(\xi)]}\,dx = \phi(\xi')\int_{\xi-\delta}^{\xi+\delta} e^{\frac{n(x-\xi)^2\,h''(\xi'')}{2}}\,dx \tag{16}$$

First let's deal with the Taylor expansion of $h(x)$ about ξ:

$$h(x) - h(\xi) = h^{'}(\xi)(x-\xi) + \frac{h^{''}(\xi^{''})}{2!}(x-\xi)^2 \tag{17}$$

Note here that the Taylor series is up to the first power and the term $\frac{h^{''}(\xi^{''})}{2!}(x-\xi)^2$ is the Lagrange form of the remainder (see[1], page 106). So there really is equality in (18), not just an approximation. This is all kosher since we know the second derivative exists (which is all we need for Taylor's theorem) and is continuous in a neigbourhood of ξ. Of course, $\xi - \delta < \xi^{''} < \xi + \delta$ in (17). Note here that if you represent $h(x) - h(\xi) = h^{'}(\xi)(x-\xi) + \frac{h^{''}(\xi^{''})}{2!}(x-\xi)^2$ + remainder you have to then show that the effect of the remainder which involves third derivatives etc is small.

Because of the assumption that $h(x)$ attains its maximum only at the point ξ in (a, b) we have that $h^{'}(\xi) = 0$ so that $h(x) - h(\xi) = \frac{h^{''}(\xi^{''})}{2!}(x-\xi)^2$ as advertised in (9).

The final point to note is why $\phi(\xi^{'})$ can be dragged out the front of the integral. Since ϕ is continuous at $x = \xi$ we can approximate the integral $\int_{\xi-\delta}^{\xi+\delta} \phi(x)\, e^{-g(x)}\, dx$ (where $g(x) > 0$ is shorthand for the exponent at issue) as follows :

$$\begin{aligned}
\Big| \int_{\xi-\delta}^{\xi+\delta} \phi(x)\, e^{-g(x)}\, dx \Big| &= \Big| \int_{\xi-\delta}^{\xi+\delta} \Big(\phi(x) - \phi(\xi) + \phi(\xi)\Big) e^{-g(x)}\, dx \Big| \\
&\leq \int_{\xi-\delta}^{\xi+\delta} \Big(\Big| \phi(x) - \phi(\xi) \Big| + \Big| \phi(\xi) \Big| \Big) e^{-g(x)}\, dx \\
&< \underbrace{2\delta e^{-g(\xi-\delta)}\epsilon}_{\text{arbitrarily small}} + |\phi(\xi)| \int_{\xi-\delta}^{\xi+\delta} e^{-g(x)}\, dx
\end{aligned} \tag{18}$$

This justifies the last line of (9) and also the statement in (10). To evaluate the integrals in (10) we need a straightforward result which is Problem 20 in ([2],page 96) which is as follows.

Let k be a positive constant and $a < \xi < b$ show that for a, b, ξ, k fixed and $n \to \infty$ that:

$$\int_a^b e^{-kn(x-\xi)^2}\, dx \sim \sqrt{\frac{\pi}{kn}} \tag{19}$$

To prove this simply let $\sqrt{kn}(x - \xi) = t$ so that $dx = \frac{dt}{\sqrt{kn}}$.

Hence:

$$\int_a^b e^{-kn(x-\xi)^2}\,dx = \frac{1}{\sqrt{kn}} \int_{-\sqrt{kn}(\xi-a)}^{\sqrt{kn}(b-\xi)} e^{-t^2}\,dt \tag{20}$$

Since $\int_{-\infty}^{\infty} e^{-t^2}\,dt = \pi$ the integral in (20) behaves aymptotically as $\sqrt{\frac{\pi}{kn}}$.

Using this result we can see that: $[\phi(\xi) - \epsilon] \int_{\xi-\delta}^{\xi+\delta} \phi(x)\, e^{\frac{n\,[(x-\xi)^2\,h''(\xi'')-\epsilon]}{2}}\,dx$ and $[\phi(\xi) + \epsilon] \int_{\xi-\delta}^{\xi+\delta} \phi(x)\, e^{\frac{n\,[(x-\xi)^2\,h''(\xi'')+\epsilon]}{2}}\,dx$ are asymptotically equal, respectively, to:

$$[\phi(\xi) - \epsilon] \sqrt{\frac{-2\pi}{n[h''(\xi) - \epsilon]}} \text{ and } [\phi(\xi) + \epsilon] \sqrt{\frac{-2\pi}{n[h''(\xi) + \epsilon]}} \tag{21}$$

Finally, since $\epsilon > 0$ is arbitrary we find that $\int_a^b \phi(x)\,[f(x)]^n\,dx \sim \phi(\xi)\,e^{nh(\xi)} \sqrt{-\frac{2\pi}{n\,h''(\xi)}}$.

Because $f(x) = e^{h(x)}$ and $h'(\xi) = 0$ and:

$$h''(x) = \frac{f(x)\,f''(x) - [f'(x)]^2}{[f(x)]^2} \quad \text{and } f'(x) = e^{h(x)}\,h'(x) \tag{22}$$

we have that $f'(\xi) = e^{h(\xi)}\,h'(\xi) = 0$.

Thus:

$$h''(\xi) = \frac{f(\xi)\,f''(\xi) - [f'(\xi)]^2}{[f(\xi)]^2} = \frac{f''(\xi)}{f(\xi)} \tag{23}$$

This means that:

$$\phi(\xi)\,e^{nh(\xi)} \sqrt{-\frac{2\pi}{n\,h''(\xi)}} = \phi(\xi)\,[f(\xi)]^{n+\frac{1}{2}} \sqrt{-\frac{2\pi}{n\,f''(\xi)}} \tag{24}$$

3 Wallis' formula

Going back to the original problem posed in (5) we have to prove that if:

$\int_0^{\frac{\pi}{2}} \sin^{2n} x\,dx = \int_0^{\frac{\pi}{2}} \cos^{2n} x\,dx = \frac{1.3....(2n-1)}{2.4....2n}\,\frac{\pi}{2}$ then:

$$\frac{1.3.\ldots(2n-1)}{2.4.\ldots 2n} \sim \frac{1}{\sqrt{n\pi}} \tag{25}$$

Clearly due to the eveness of the function:

$$2\int_0^{\frac{\pi}{2}} \sin^{2n} x\, dx = \int_0^{\pi} \sin^{2n} x\, dx = \frac{1.3.\ldots.(2n-1)}{2.4.\ldots.2n}\,\pi \tag{26}$$

If we let $a = 0$, $b = \pi$, $\phi(x) = \frac{1}{\pi}$ and $f(x) = \sin^2 x$, $\xi = \frac{\pi}{2}$ in Laplace's method we can perform the asymptotic estimate. However, we have to check that the hypotheses are satisfied. In this case $h(x) = \ln\sin^2 x$ so that the maximum only occurs at $\xi = \frac{\pi}{2}$. Note that $h^{''}(x) = -2\csc^2 x$ so that $h^{''}(\xi) = -2 < 0$ and $f^{''}(x) = -2\sin^2 x + 2\cos^2 x$ so that $f^{''}(\xi) = -2$. Hence the hypotheses of Laplace's method are satisfied:

$$\int_0^{\frac{\pi}{2}} \sin^{2n} x\, dx \sim \frac{1}{\pi}\, \sin^{2n+1}(\frac{\pi}{2})\sqrt{\frac{-2\pi}{-2n}} = \sqrt{\frac{1}{n\pi}} \tag{27}$$

4 An application to Stirling's formula

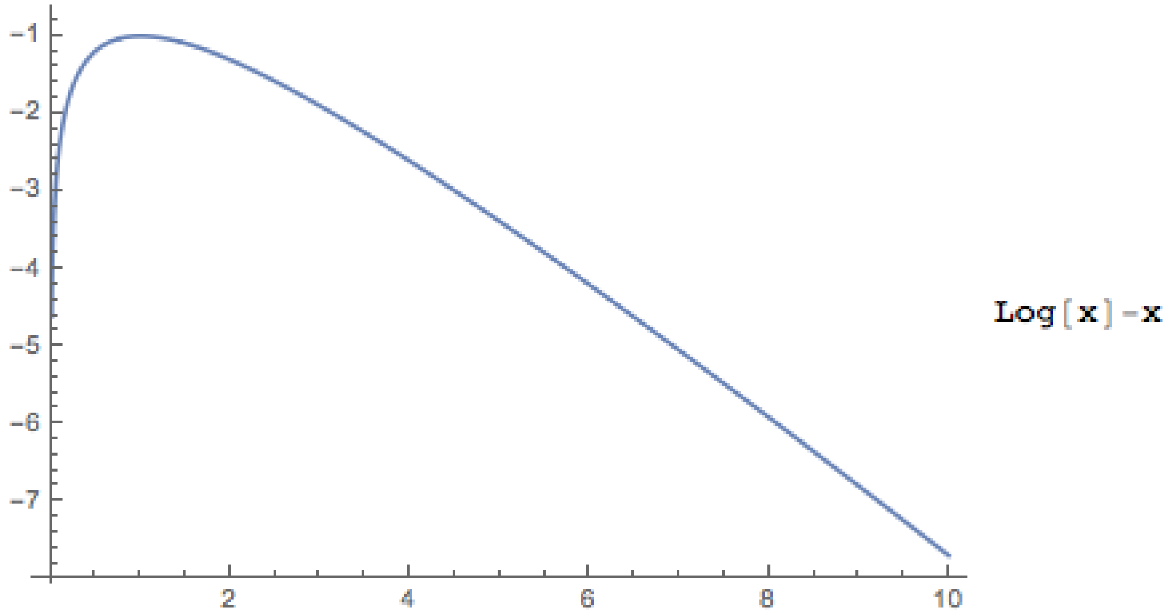
Problem 205 in ([2], page 97) involves using Laplace's method to show that for n positive, $n \to \infty$:

$$\Gamma(n+1) = \int_0^{\infty} e^{-x}x^n\, dx \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n} \tag{28}$$

Make the substitution$x = ny$ in the integral in (28) giving:

$$\Gamma(n+1) = n^{n+1}\int_0^{\infty} (e^{-x}x)^n\, dx \tag{29}$$

For Laplace's method we have: $a = 0$, $b = \infty$, $\phi(x) = 1$ and $f(x) = e^{-x}x$, $\xi = 1$. Note that in this case $e^{-x}x = e^{\ln x - x} = e^{h(x)}$ so that $h(x) = \ln x - x$. Also $h^{'}(x) = \frac{1}{x} - 1$ and $h^{''}(x) = \frac{-1}{x^2}$. There is a global maximum at $\xi = 1$ - see the graph below:



Finally, $f^{''}(x) = (x-2)\,e^{-x}$ so that $f^{''}(\xi) = \frac{-1}{e}$ and $f(1) = \frac{1}{e}$. Plugging these values into Laplace's method we have:

$$\Gamma(n+1) \sim n^{n+1}\phi(1)[f(1)]^{n+\frac{1}{2}}\sqrt{\frac{2\pi e}{n}} = \left(\frac{n}{e}\right)^n\sqrt{2\pi n} \tag{30}$$

5 Final comments

You will see proofs of Laplace's theorem which run something like this. In what follows it is assumed that g assumes a strict minimum (ie $-g$ is a strict maximum) at an interior critical point c and $g^{'}(c) = 0$, $g^{''}(c) > 0$ and $f(c) \neq 0$. For $\lambda \gg 1$:

$$
\begin{aligned}
I(\lambda) = \int_a^b f(t)\, e^{-\lambda g(t)}\, dt = e^{-\lambda g(c)} \int_a^b f(t)\, e^{-\lambda [g(t)-g(c)]}\, dt \\
\approx e^{-\lambda g(c)} \int_{c-\epsilon}^{c+\epsilon} f(t)\, e^{-\lambda [g(t)-g(c)]}\, dt \\
\approx e^{-\lambda g(c)} f(c) \int_{c-\epsilon}^{c+\epsilon} e^{-\lambda [g(t)-g(c)]}\, dt \\
\approx e^{-\lambda g(c)} f(c) \int_{c-\epsilon}^{c+\epsilon} e^{-\lambda [g^{'}(c)(t-c)+\frac{1}{2} g^{''}(c)(t-c)^2]}\, dt \\
\approx e^{-\lambda g(c)} f(c) \int_{c-\epsilon}^{c+\epsilon} e^{\frac{-\lambda}{2} g^{''}(c)(t-c)^2}\, dt \\
\approx e^{-\lambda g(c)} f(c) \int_{-\infty}^{\infty} e^{\frac{-\lambda}{2} g^{''}(c)(t-c)^2}\, dt \\
= e^{-\lambda g(c)} f(c) \int_{-\infty}^{\infty} e^{\frac{-\lambda}{2} g^{''}(c)\, s^2}\, ds \\
= e^{-\lambda g(c)} f(c) \sqrt{\frac{2\pi}{\lambda g''(c)}}
\end{aligned}
\tag{31}
$$

The precise justifications for each step are usually not given, but the recipe in such proofs is clearly consistent with what is set out above.

6 References

1. David Bressoud, "A Radical Approach to Real Analysis, Second Edition", The Mathematical Association of America, 2007.
2. George Polya and Gabor Szegö, "Problems and Theorems in Analysis I", Reprint of the 1978 edition, Springer, 1998.

7 History

Created 08/06/2015
Updated 27/06/2015 - fixed some pesky typos. 28/09/2015 - corrected a typo