Littlewood's proof of the Hölder and Minkowski inequalities

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J E Littlewood (jokingly referred to by some as Hardy's "wife" because of the number of books and articles they collaborated on) wrote a book titled "Lectures on the Theory of Functions" [1] which deals in detail with many aspects of integration theory among other things. Fundamental tools in the theory are the Hölder and Minkowski inequalities. Of the two inequalities, Hölder's is the more fundamental since it is used to prove Minkowski's inequality.

There are various ways Hölder's inequality can be proved and interested readers should refer to Chapter 9 of J. Michael Steele's book [2] for further information. Copson, for example, starts out with $= \log x$ where $x > 0$ and uses downward concavity to advantage ([2] pp 26-28). Every student of analysis has to know both inequalities and how they are proved. You simply cannot do any serious analysis without them.

Hölder's inequality [H]

$$\boxed{\sum_{j=1}^{n} a_j b_j \le (\sum_{j=1}^{n} a_j^p)^{\frac{1}{p}} (\sum_{j=1}^{n} b_j^q)^{\frac{1}{q}}} \tag{1}$$

where $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $a, b > 0$. If either a or b is zero H is obviously and uninterestingly true. Note that when $p = 2$ we get Cauchy's inequality.

Hölder's inequality underpins "mean value" results such as this:

$$\frac{1}{2\pi}\Big|\int_{-\pi}^{\pi} fg\, d\theta\Big| \le \Big(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f|^p\, d\theta\Big)^{\frac{1}{p}} \Big(\frac{1}{2\pi}\int_{-\pi}^{\pi} |g|^q\, d\theta\Big)^{\frac{1}{q}} \tag{2}$$

Minkowski's inequality [M]

$$\boxed{\Big(\sum_{j=1}^{n} (a_j + b_j)^k\Big)^{\frac{1}{k}} \le \Big(\sum_{j=1}^{n} a_j^k\Big)^{\frac{1}{k}} + \Big(\sum_{j=1}^{n} b_j^k\Big)^{\frac{1}{k}}} \tag{3}$$

for $k \geq 1$.

This inequality underpins results such as this:

$$\left(\int_E \left(|f_1(x) + f_2(x)|\right)^k dx \right)^{\frac{1}{k}} \leq \left(\int_E \left(|f_1(x)|\right)^k dx \right)^{\frac{1}{k}} + \left(\int_E \left(|f_2(x)|\right)^k dx \right)^{\frac{1}{k}} \quad (4)$$

1 Proof of Hölder's inequality

We first start with some definitions with the summation subscript suppressed for convenience:

$$\begin{aligned} U^p &= \sum a^p \\ V^q &= \sum b^q \\ W &= \sum ab \\ x &= a^{\frac{1}{q}} b^{-\frac{1}{p}} \end{aligned} \quad (5)$$

The crux of the proof is to construct a ratio as follows in the variable x and use calculus to show that this function has a minimum that allows us to assert (1). (Note that in [1] there are some typographical errors on page 4).

$$\begin{aligned} t(x) &= \frac{\frac{a^p}{p} + \frac{b^q}{q}}{ab} \\ &= \frac{x^p}{p} + \frac{x^{-q}}{q} \end{aligned} \quad (6)$$

We need to check (6):

$$\begin{aligned} \frac{x^p}{p} + \frac{x^{-q}}{q} &= \frac{\left(a^{\frac{1}{q}} b^{-\frac{1}{p}}\right)^p}{p} + \frac{\left(a^{\frac{1}{q}} b^{-\frac{1}{p}}\right)^{-q}}{q} \\ &= \frac{a^{\frac{p}{q}} b^{-1}}{p} + \frac{a^{-1} b^{\frac{q}{p}}}{q} \\ &= \frac{qa^{\frac{p}{q}+1} + pb^{\frac{q}{p}+1}}{pqab} \\ &= \frac{qa^{\frac{pq}{q}} + pb^{\frac{pq}{p}}}{pqab} \\ &= \frac{qa^p + pb^q}{pqab} \\ &= \frac{\frac{a^p}{p} + \frac{b^q}{q}}{ab} \end{aligned} \quad (7)$$

Note here that since $\frac{1}{p}+\frac{1}{q}=1$ we have $p+q=pq$ and also that $(p-1)(q-1)=1$.

Now we can differentiate $t(x)$ and we get:

$$t^{'}(x)=x^{p-1}-x^{-(q+1)} \tag{8}$$

which equals 0 when $x=1$.

For $x>0$ we have a stationary point at $x=1$ and from (8) the second derivative is:

$$t^{''}(x)=(p-1)x^{p-2}+(q+1)x^{-(q+2)} \tag{9}$$

$$t^{''}(1)=p-1+q+1=p+q>0 \tag{10}$$

So we have a minimum for $t(x)$ at $x=1$. The minimum is $t(1)=\frac{1}{p}+\frac{1}{q}=1$. Hence from (6):

$$ab \leq \frac{a^p}{p}+\frac{b^q}{q} \tag{11}$$

To get to (1) we need to bring sums into the picture. For $\lambda>0$ we have:

$$ab=\lambda a\,\lambda^{-1}b \leq \frac{\lambda^p a^p}{p}+\frac{\lambda^{-q}b^q}{q} \tag{12}$$

Hence

$$\begin{aligned}\sum ab &\leq \sum\frac{\lambda^p a^p}{p}+\sum\frac{\lambda^{-q}b^q}{q}\\ \therefore W &\leq \lambda^p\frac{U^p}{p}+\lambda^{-q}\frac{V^q}{q}\end{aligned} \tag{13}$$

In (1) we can suppose that $U,V>0$ so we can choose λ so that:

$$\lambda^p U^p=\lambda^{-q}V^q \tag{14}$$

Hence:

$$\begin{aligned}\left(\lambda^p U^p\right)^{\frac{1}{p}} &=\left(\lambda^{-q}V^q\right)^{\frac{1}{p}}\\ &=\left(\lambda^{-q}V^q\right)^{1-\frac{1}{q}}\\ \therefore \left(\lambda^p U^p\right)^{\frac{1}{p}}\left(\lambda^{-q}V^q\right)^{\frac{1}{q}} &=\lambda^{-q}V^q=\lambda^p U^p\end{aligned} \tag{15}$$

The last line of (15) says that:

$$UV = \lambda^p U^p = \lambda^{-q} V^q \tag{16}$$

Therefore using (13) we have:

$$W \leq \frac{UV}{p} + \frac{UV}{q} = UV(\frac{1}{p} + \frac{1}{q}) = UV \tag{17}$$

In other words using (5) we get what we wanted:

$$\sum ab \leq \left(\sum a^p\right)^{\frac{1}{p}} \left(\sum b^q\right)^{\frac{1}{q}} \tag{18}$$

2 Proof of Minkowski's inequality

Minkowski's inequality is uninterestingly true when $k = 1$ since we simply have:

$$\sum_{j=1}^{n}(a_j + b_j) \leq \sum_{j=1}^{n} a_j + \sum_{j=1}^{n} b_j \tag{19}$$

So we suppose $k > 1$. Then (again summation indices are suppressed) we let $M^k = \sum(a+b)^k$ and $\frac{1}{k} + \frac{1}{k'} = 1$ so that:

$$\begin{aligned}
M^k &= \sum (a+b)^k \\
&= \sum \left((a+b)^{k-1}(a+b)\right) \\
&= \sum (a+b)^{k-1} a + \sum (a+b)^{k-1} b \\
&= \sum (aB) + \sum (Ab) \quad \text{ where } A = B = (a+b)^{k-1} \\
&\leq \left(\sum a^k\right)^{\frac{1}{k}} \left(\sum B^{k'}\right)^{\frac{1}{k'}} + \left(\sum A^k\right)^{\frac{1}{k}} \left(\sum b^{k'}\right)^{\frac{1}{k'}} \\
&= \left(\sum a^k\right)^{\frac{1}{k}} \left(\sum (a+b)^{(k-1)k'}\right)^{\frac{1}{k'}} + \left(\sum (a+b)^{k(k-1)}\right)^{\frac{1}{k}} \left(\sum b^{k'}\right)^{\frac{1}{k'}} \\
&= \left(\sum a^k\right)^{\frac{1}{k}} \left(\sum (a+b)^k\right)^{\frac{1}{k'}} + \left(\sum (a+b)^k\right)^{\frac{1}{k'}} \left(\sum b^{k'}\right)^{\frac{1}{k'}} \\
&= \left(\sum (a+b)^k\right)^{\frac{1}{k'}} \left[\left(\sum a^k\right)^{\frac{1}{k}} + \left(\sum b^{k'}\right)^{\frac{1}{k'}}\right] \\
&= M^{\frac{k}{k'}} \left[\left(\sum a^k\right)^{\frac{1}{k}} + \left(\sum b^{k'}\right)^{\frac{1}{k'}}\right] \quad \text{ since } M^{\frac{k}{k'}} = \left(\sum (a+b)^k\right)^{\frac{1}{k'}} \\
\therefore M^{\frac{k(k'-1)}{k'}} &= M^{\frac{k'}{k'}} = M \leq \left(\sum a^k\right)^{\frac{1}{k}} + \left(\sum b^{k'}\right)^{\frac{1}{k'}}
\end{aligned} \tag{20}$$

and so [M] is proved.

Littlewood extends [H] and [M] in a multitude of ways which you can check out for yourself by consulting the reference.

3 References

[1] J E Littlewood, "Lectures on the Theory of Functions", Oxford University Press, 1944. The scanned version can be found here: `https://ia801603.us.archive.org/3/items/in.ernet.dli.2015.205725/2015.205725.Lectures-In.pdf`

[2] J Michael Steele, "The Cauchy-Schwarz Master Class", Cambridge University Press, 2004

[3] E T Copson, "Metric Spaces", Cambridge University Press, 1968

4 History

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