Matrix exponentiation - some solved Tripos problems

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November 20, 2019

1 Background

As a vehicle for explaining matrix exponentiation it is useful to actually apply the theory to some Tripos Part 1A problems from May 2019. In order to do these problems some basic theory is required.

First, the concept of the exponential of a square matrix M is defined as follows:

$$e^M = I + \sum_{n=1}^{\infty} \frac{M^n}{n!} \tag{1}$$

I is the identity matrix.

Clearly, (1) requires concepts of convergence so that the infinite sum makes sense. A matrix sequence $\{A_k\}$ where $A_k \in \mathbb{C}^{n \times n}$ is said to converge if it converges elementwise, ie $[A_k]_{ij}$ (this denotes the i^{th} row and j^{th} column of the matrix at issue) converges as $k \to \infty$ for all i and j. As usual, a matrix series is said to converge if the corresponding (matrix) sequence of partial sums converges. In the Tripos problems which follow students were told that it was not necessary to consider issues of convergence - indeed the matrices in the problems in fact satisfy convergence requirements but that is not the focus of the problems.

Secondly we need some or all of the following properties which are proved in the theory of matrix exponentiation (the Appendix contains some proofs):

$$\left(e^A\right)^{-1} = e^{-A} \tag{2}$$

$$e^A\, e^B = e^{A+B} = e^B\, e^A \text{ if } AB = BA \tag{3}$$

$$e^{tA}\, e^{sA} = e^{(t+s)A} \tag{4}$$

$$e^{TBT^{-1}} = Te^B T^{-1} \tag{5}$$

$$\frac{d}{dt} e^{tA} = Ae^{tA} \tag{6}$$

$$e^A\, B = B\, e^A \text{ if } AB = BA \tag{7}$$

2 Problem 8A (a)

Calculate the elements of R and S where θ is a real number:

$$R = e^M \tag{8}$$

$$S = e^N \tag{9}$$

and

$$M = \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix} \tag{10}$$

$$N = \begin{pmatrix} 0 & \theta \\ \theta & 0 \end{pmatrix} \tag{11}$$

You are told that the answer must be in closed form ie not given as a series.

2.1 Solution to Problem 8A(a)

One straightforward line of attack is to diagonalise M ie $M = P\Omega P^{-1}$. We have that:

$$|M - \lambda I| = \lambda^2 + \theta^2 = 0 \tag{12}$$

which gives us the two eigenvalues $\lambda = \pm i\,\theta$.

The eigenvector corresponding to $\lambda = i\theta$ solves:

$$\begin{pmatrix} -i\theta & -\theta \\ \theta & -i\theta \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{13}$$

so $\vec{u_1} = \begin{pmatrix} i \\ 1 \end{pmatrix}$ is a suitable eigenvector.

The eigenvector corresponding to $\lambda = -i\theta$ solves:

$$\begin{pmatrix} i\theta & -\theta \\ \theta & i\theta \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{14}$$

so $\vec{u_2} = \begin{pmatrix} 1 \\ i \end{pmatrix}$ is a suitable eigenvector.

Hence:

$$P = \begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix} \tag{15}$$

and

$$\Omega = \begin{pmatrix} i\theta & 0 \\ 0 & -i\theta \end{pmatrix} \tag{16}$$

and

$$P^{-1} = \begin{pmatrix} -\frac{i}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{i}{2} \end{pmatrix} \tag{17}$$

Thus we have:

$$\begin{aligned} M^n =& P\Omega^n P^{-1} \\ =& \begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix} \begin{pmatrix} (i\theta)^n & 0 \\ 0 & (-i\theta)^n \end{pmatrix} \begin{pmatrix} -\frac{i}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{i}{2} \end{pmatrix} \\ =& \begin{pmatrix} i(i\theta)^n & (-i\theta)^n \\ (i\theta)^n & i(-i\theta)^n \end{pmatrix} \begin{pmatrix} -\frac{i}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{i}{2} \end{pmatrix} \\ =& \begin{pmatrix} \frac{(i\theta)^n + (-i\theta)^n}{2} & \frac{i(i\theta)^n - i(-i\theta)^n}{2} \\ \frac{-i(i\theta)^n + i(-i\theta)^n}{2} & \frac{(i\theta)^n + (-i\theta)^n}{2} \end{pmatrix} \end{aligned} \tag{18}$$

Finally:

$$
\begin{aligned}
e^M &= \begin{pmatrix} \sum_{n=0}^{\infty} \frac{(i\theta)^n + (-i\theta)^n}{2n!} & \sum_{n=0}^{\infty} \frac{i(i\theta)^n - i(-i\theta)^n}{2n!} \\ \sum_{n=0}^{\infty} \frac{-i(i\theta)^n + i(-i\theta)^n}{2n!} & \sum_{n=0}^{\infty} \frac{(i\theta)^n + (-i\theta)^n}{2n!} \end{pmatrix} \\
&= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}
\end{aligned} \tag{19}
$$

Thankfully Stephen Wolfram agrees with me:

```
In[2]:= MatrixExp[( 0  -θ
                    θ   0 )] // TraditionalForm
```

Out[2]//TraditionalForm=

$$
\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}
$$

We now calculate $S = e^N$ (see (11)). To do this we will use another technique. One can quickly establish the following structure of N^k by some simple multiplications and an implicit appeal to induction:

$$
N^{2k+1} = \begin{pmatrix} 0 & \theta^{2k+1} \\ \theta^{2k+1} & 0 \end{pmatrix} \tag{20}
$$

$$
N^{2k} = \begin{pmatrix} \theta^{2k} & 0 \\ 0 & \theta^{2k} \end{pmatrix} \tag{21}
$$

Hence we have:

$$
\begin{aligned}
e^N &= \sum_{k=0}^{\infty} \left[\frac{1}{(2k+1)!} \begin{pmatrix} 0 & \theta^{2k+1} \\ \theta^{2k+1} & 0 \end{pmatrix} + \frac{1}{(2k)!} \begin{pmatrix} \theta^{2k} & 0 \\ 0 & \theta^{2k} \end{pmatrix} \right] \\
&= \begin{pmatrix} \sum_{k=0}^{\infty} \frac{\theta^{2k}}{(2k)!} & \sum_{k=0}^{\infty} \frac{\theta^{2k+1}}{(2k+1)!} \\ \sum_{k=0}^{\infty} \frac{\theta^{2k+1}}{(2k+1)!} & \sum_{k=0}^{\infty} \frac{\theta^{2k}}{(2k)!} \end{pmatrix} \\
&= \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix}
\end{aligned} \tag{22}
$$

Note that if you go through and diagonalise N you will wind up with this:

$$\begin{pmatrix} \frac{e^\theta+e^{-\theta}}{2} & \frac{e^\theta-e^{-\theta}}{2} \\ \frac{e^\theta-e^{-\theta}}{2} & \frac{e^\theta+e^{-\theta}}{2} \end{pmatrix} = \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix} \tag{23}$$

Again, Mr Wolfram agrees:

In[3]:= $\texttt{MatrixExp}\left[\begin{pmatrix} 0 & \theta \\ \theta & 0 \end{pmatrix}\right]$ `// TraditionalForm`

Out[3]//TraditionalForm=

$$\begin{pmatrix} \frac{e^{-\theta}}{2} + \frac{e^{\theta}}{2} & \frac{e^{\theta}}{2} - \frac{e^{-\theta}}{2} \\ \frac{e^{\theta}}{2} - \frac{e^{-\theta}}{2} & \frac{e^{-\theta}}{2} + \frac{e^{\theta}}{2} \end{pmatrix}$$

3 Problem 8A(b)

Show that $RR^T = I$ and that:

$$SJS = J \tag{24}$$

where $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

3.1 Solution to Problem 8A(b)

For any two square matrices A, B we have:

$$(A+B)^T = A^T + B^T \tag{25}$$

and

$$(AB)^T = B^T A^T \tag{26}$$

(25) generalises to infinite sums and from (26) we see that:

$$(M^2)^T = M^T M^T = (M^T)^2 \tag{27}$$

and so inductively we have that:

$$(M^k)^T = (M^T)^k \tag{28}$$

With these ingredients we see that:

$$R^T = (e^M)^T = \Big(\sum_{k=0}^{\infty} \frac{M^k}{k!}\Big)^T = \sum_{k=0}^{\infty} \frac{(M^k)^T}{k!} = \sum_{k=0}^{\infty} \frac{(M^T)^k}{k!} = e^{M^T} \tag{29}$$

Now $M^T = -M$ and because M and M commute M^T and M commute, so using (3) we have:

$$RR^T = e^M e^{-M} = e^O = I \tag{30}$$

To prove (24) recall that we have established in (22) that $S = e^N = \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix}$. All we need to do is a simple multiplication:

$$\begin{aligned} SJS =& \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix} \\ =& \begin{pmatrix} \cosh\theta & -\sinh\theta \\ \sinh\theta & -\cosh\theta \end{pmatrix} \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix} \\ =& \begin{pmatrix} \cosh^2\theta - \sinh^2\theta & 0 \\ 0 & -(\cosh^2\theta - \sinh^2\theta) \end{pmatrix} \\ =& \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ =& J \end{aligned} \tag{31}$$

4 Problem 8A(c)

Consider the matrices:

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \tag{32}$$

and

$$B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{33}$$

Calculate:
(i) e^{xA};
(ii) e^{xB}

The answers must be in closed form.

4.1 Solution to (i)

The main insight to solving this problem is that A contains $M_{\frac{1}{2}} = \begin{pmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$ as a block matrix and the structure of A is maintained through successive matrix multiplications (you can actually prove this if pushed but it is pretty obvious) and is simply a consequence of the null column/row of A. We also know from (18) what the components of M^k are and that enables us to solve this problem without any real calculation.

$$A = \begin{pmatrix} 0 & 0 \\ 0 & M_{1/2} \end{pmatrix} \longrightarrow A^k = \begin{pmatrix} 0 & 0 \\ 0 & M_{1/2}^k \end{pmatrix}$$

$$
\begin{aligned}
e^{(xA)} &= I + \sum_{k=1}^{\infty} \frac{(xA)^k}{k!} \\
&= I + \begin{pmatrix} 0 & 0 & 0 \\ 0 & \displaystyle\sum_{k=1}^{\infty} \frac{(i\frac{x}{2})^k + (-i\frac{x}{2})^k}{2k!} & \displaystyle\sum_{k=1}^{\infty} \frac{i(i\frac{x}{2})^k - i(-i\frac{x}{2})^k}{2k!} \\ 0 & \displaystyle\sum_{k=1}^{\infty} \frac{-i(i\frac{x}{2})^k + i(-i\frac{x}{2})^k}{2k!} & \displaystyle\sum_{k=1}^{\infty} \frac{(i\frac{x}{2})^k + (-i\frac{x}{2})^k}{2k!} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\frac{x}{2} & -\sin\frac{x}{2} \\ 0 & \sin\frac{x}{2} & \cos\frac{x}{2} \end{pmatrix}
\end{aligned}
\tag{34}
$$

Mathematica verifies this result:

```
In[•]:= A = {{0, 0, 0}, {0, 0, -x/2}, {0, x/2, 0}};

In[•]:= MatrixExp[A] // TraditionalForm
```

Out[•]//TraditionalForm=

$$
\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\left(\frac{x}{2}\right) & -\sin\left(\frac{x}{2}\right) \\ 0 & \sin\left(\frac{x}{2}\right) & \cos\left(\frac{x}{2}\right) \end{pmatrix}
$$

5 Solution to (ii)

Ignoring the hints in the prior problem you can simply diagonalise B and when you do this you get:

$$
B = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}
\tag{35}
$$

Hence:

$$
\begin{aligned}
(xB)^k &= \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & x^k & 0 \\ 0 & 0 & (-x)^k \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \\
&= \begin{pmatrix} 0 & x^k & -(-x)^k \\ 0 & 0 & 0 \\ 0 & x^k & (-x)^k \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{2}(x^k + (-x)^k) & 0 & \frac{1}{2}(x^k - (-x)^k) \\ 0 & 0 & 0 \\ \frac{1}{2}(x^k - (-x)^k) & 0 & \frac{1}{2}(x^k + (-x)^k) \end{pmatrix}
\end{aligned}
\tag{36}
$$

Using (23) what we ultimately end up with is (remember $e^{(xB)} = I + \sum_{k=1}^{\infty} \frac{(xB)^k}{k!}$):

$$
e^{(xB)} = \begin{pmatrix} \cosh x & 0 & \sinh x \\ 0 & 1 & 0 \\ \sinh x & 0 & \cosh x \end{pmatrix} \tag{37}
$$

But what they wanted you to do was to apply the same logic as in part (i) to notice that effectively B is N (see (11) with $\theta = 1$) with null column/rows embedded in it. As we have already solved what happens to powers of N (see (20)-(21)) there is no real calculation to do other than to confirm the structural behaviour of B^k. The following diagrams should help, noting that we use the columns of B to multipy rows of B - making old Gil Strang a happy man:

$$B = \begin{array}{c} \begin{array}{ccc} \vec{c}_1 & \vec{c}_2 & \vec{c}_3 \end{array} \\ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \end{array}$$

$$B^2 = \left[\vec{c}_1, \vec{c}_2, \vec{c}_3\right] \begin{bmatrix} \vec{c}_1^{\,T} \\ \vec{c}_2^{\,T} \\ \vec{c}_3^{\,T} \end{bmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$B^3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} (1\ 0\ 0) + \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} (0\ 0\ 0) + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} (0\ 0\ 1)$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$B^4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{etc.}$$

$$B^{2k} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad B^{2k+1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

This is the same structure that we had with powers of N - see (20)-(21). The result in (37) then follows without any effort.

Again, Mr Wolfram agrees:

In[10]:= `B =` $\begin{pmatrix} 0 & 0 & x \\ 0 & 0 & 0 \\ x & 0 & 0 \end{pmatrix}$;

In[11]:= `MatrixExp[B] // TraditionalForm`

Out[11]//TraditionalForm=

$$\begin{pmatrix} \frac{e^{-x}}{2}+\frac{e^{x}}{2} & 0 & \frac{e^{x}}{2}-\frac{e^{-x}}{2} \\ 0 & 1 & 0 \\ \frac{e^{x}}{2}-\frac{e^{-x}}{2} & 0 & \frac{e^{-x}}{2}+\frac{e^{x}}{2} \end{pmatrix}$$

which simplifies to (37).

6 Problem 8A (d)

Defining

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{38}$$

find the elements of the following matrices, where N is a natural number. The answers must be in closed form.

(i) $\sum_{n=1}^{N} \left[e^{(xA)} C \left(e^{(xA)} \right)^{T} \right]^{n}$

(ii) $\sum_{n=1}^{N} \left[e^{(xB)} C \left(e^{(xB)} \right) \right]^{n}$

6.1 Solution to (i)

$$\begin{aligned}
\sum_{n=1}^{N}\left[e^{(xA)}C\left(e^{(xA)}\right)^{T}\right]^{n} &= \sum_{n=1}^{N}\left[\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\frac{x}{2} & -\sin\frac{x}{2}\\ 0 & \sin\frac{x}{2} & \cos\frac{x}{2}\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\frac{x}{2} & -\sin\frac{x}{2}\\ 0 & \sin\frac{x}{2} & \cos\frac{x}{2}\end{pmatrix}^{T}\right]^{n}\\
&= \sum_{n=1}^{N}\left[\begin{pmatrix}1 & 0 & 0\\ 0 & -\cos\frac{x}{2} & \sin\frac{x}{2}\\ 0 & -\sin\frac{x}{2} & -\cos\frac{x}{2}\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\frac{x}{2} & \sin\frac{x}{2}\\ 0 & -\sin\frac{x}{2} & \cos\frac{x}{2}\end{pmatrix}\right]^{n}\\
&= \sum_{n=1}^{N}\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{pmatrix}^{n}\\
&= \sum_{n=1}^{N}\begin{pmatrix}1 & 0 & 0\\ 0 & (-1)^{n} & 0\\ 0 & 0 & (-1)^{n}\end{pmatrix}\\
&= \begin{pmatrix}n & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} \text{ if } n \text{ is even}\\
&= \begin{pmatrix}n & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{pmatrix} \text{ if } n \text{ is odd}
\end{aligned} \tag{39}$$

6.2 Solution to (ii)

$$\begin{aligned}
\sum_{n=1}^{N}\left[e^{(xB)}C\Big(e^{(xB)}\Big)\right]^n &= \sum_{n=1}^{N}\left[\begin{pmatrix}\cosh x & 0 & \sinh x\\ 0 & 1 & 0\\ \sinh x & 0 & \cosh x\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{pmatrix}\begin{pmatrix}\cosh x & 0 & \sinh x\\ 0 & 1 & 0\\ \sinh x & 0 & \cosh x\end{pmatrix}\right]^n\\
&= \sum_{n=1}^{N}\left[\begin{pmatrix}\cosh x & 0 & -\sinh x\\ 0 & -1 & 0\\ \sinh x & 0 & -\cosh x\end{pmatrix}\begin{pmatrix}\cosh x & 0 & \sinh x\\ 0 & 1 & 0\\ \sinh x & 0 & \cosh x\end{pmatrix}\right]^n\\
&= \sum_{n=1}^{N}\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{pmatrix}^n\\
&= \sum_{n=1}^{N}\begin{pmatrix}1 & 0 & 0\\ 0 & (-1)^n & 0\\ 0 & 0 & (-1)^n\end{pmatrix}\\
&= \begin{pmatrix}n & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} \text{ if } n \text{ is even}\\
&= \begin{pmatrix}n & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{pmatrix} \text{ if } n \text{ is odd}
\end{aligned} \tag{40}$$

7 Appendix

We start with a proof of (3).

$$\begin{aligned}
e^A\,e^B &= \Big(\sum_{k=0}^{\infty}\frac{A^k}{k!}\Big)\Big(\sum_{k=0}^{\infty}\frac{B^k}{k!}\Big)\\
&= \sum_{k=0}^{\infty}\sum_{j=0}^{k}\frac{A^j}{j!}\,\frac{B^{k-j}}{(k-j)!}\\
&= \sum_{k=0}^{\infty}\sum_{j=0}^{k}\frac{1}{k!}\binom{k}{j}A^jB^{k-j}\\
&= \sum_{k=0}^{\infty}(A+B)^k\\
&= e^{A+B}
\end{aligned} \tag{41}$$

Because A and B commute the sums in (41) can be thought of as just real numbers and so we can also say that $e^A\, e^B = e^{A+B} = e^B\, e^A$. Note that the product of two series is as follows:

$$\begin{aligned}\Big(\sum_{k=0}^{n} u_k\Big)\Big(\sum_{k=0}^{n} v_k\Big) &= \sum_{k=0}^{n}(u_0v_k + u_1v_{k-1} + \cdots + u_kv_0)\\ &= \sum_{k=0}^{n}\sum_{j=0}^{k} u_jv_{k-j}\end{aligned} \tag{42}$$

The proof of (2) is now easy using (3) since when $B = -A$ we have that $e^A\, e^{-A} = e^{A-A} = e^O = I$ hence $\left(e^A\right)^{-1} = e^{-A}$.

Similarly, (4) follows from, (3) since tA and sA commute.

To prove (5) we note that:

$$\begin{aligned}(TBT^{-1})^k =& \underbrace{(TBT^{-1})\,(TBT^{-1})\ldots(TBT^{-1})}_{\text{k terms}}\\ =& TB(T^{-1}T)\,B(T^{-1}T)\,\ldots(T^{-1}T)BT^{-1}\\ =& TB^kT^{-1}\end{aligned} \tag{43}$$

From this it follows that:

$$\sum_{k=0}^{\infty}\frac{(TBT^{-1})^k}{k!} = \sum_{k=0}^{\infty}\frac{TB^kT^{-1}}{k!} = Te^BT^{-1} \tag{44}$$

To prove (7) we note that because $AB = BA$:

$$\begin{aligned}A^kB =& A^{k-1}AB\\ =& A^{k-1}BA\\ =& A^{k-2}ABA\\ =& A^{k-2}BA^2\\ =& A^{k-3}ABA^2\\ =& A^{k-3}BA^3\\ &\vdots\\ =& BA^k\end{aligned} \tag{45}$$

Thus we have:

$$\lim_{n\to\infty}\Big(\sum_{k=0}^{n}\frac{A^k}{k!}\Big)B = B\lim_{n\to\infty}\Big(\sum_{k=0}^{n}\frac{A^k}{k!}\Big) \tag{46}$$

proving (7).

For the purposes of these problems one does not need to use (6) so I can't be bothered proving it. Every differential equations course will have a proof of this fundamental property.

8 References

Cambridge Mathematical Tripos Part 1A, 30 May 2019
`https://www.maths.cam.ac.uk/undergrad/pastpapers/files/2019/paperia_1_2019.pdf`

9 History

Created

20 November 2019