Mean square approximation of functions with discontinuities

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1 Background

Series expansions of functions is a fundamental concept in mathematics and mathematical physics. Representing a function by a linear combination of simpler components is of course the foundation of Fourier theory, wavelet theory and much more. The intrinsic issue with such processes is how close the linear combination is to the function under consideration. For instance, the Weierstrass approximation theorem shows that one can take a continuous function on $[a, b]$ and approximate it uniformly by polynomials on that interval. The set of polynomials $1, x, x^2, x^3, \dots$ form a complete system of functions which are building blocks for a continuous function on closed intervals. Note that the Weierstrass approximation theorem gives you not just convergence in the mean (to be explained below) but uniform convergence. One can find a proof of the Weierstrass approximation theorem in any analysis textbook but for mathematical physicists the proof in Hilbert and Courant's "Methods of Mathematical Physics Volume 1" is rigorous and easy to follow ([1], pages 65-68)

In what follows we consider piecewise continuous functions and examine mean square convergence having regard to a finite number of discontinuities.

2 Orthonormal systems

We start with an orthonormal system $\phi_1, \phi_2, \dots$. Such a system is orthogonal and normalised to 1. If we have a real valued continuous function f the components of f with respect the orthonormal system $\{\phi_i\}$ are:

$$c_\nu = (f, \phi_\nu) = \int_a^b f(x)\phi_\nu(x)\,dx \tag{1}$$

In what follows the limits of the integral will be omitted for convenience.

Since the system is orthonormal we have that:

$$(\phi_\nu, \phi_\mu) = \delta_{\nu\,\mu} = \begin{cases} 1 & \text{if } \nu = \mu \\ 0 & \text{if } \nu \neq \mu \end{cases} \tag{2}$$

The norm of the real valued function f is:

$$N(f) = \|f\| = (f, f) = \int f^2 \, dx \tag{3}$$

In the case of complex valued functions the norm is:

$$N(f) = \|f\| = (f, \bar{f}) = \int f\bar{f} \, dx \tag{4}$$

The orthogonality relation for real valued functions is:

$$(f, g) = 0 \tag{5}$$

And for complex valued functions the orthogonality relation is:

$$(f, \bar{g}) = (\bar{f}, g) = 0 \tag{6}$$

The Cauchy-Schwarz inequality is:

$$(f, g)^2 \leq (f, f)(g, g) \tag{7}$$

A classic example of a complex orthonormal system is from Fourier theory, namely, this system of exponentials on $[0, 2\pi]$:

$$\frac{1}{\sqrt{2\pi}}, \frac{e^{ix}}{\sqrt{2\pi}}, \frac{e^{2ix}}{\sqrt{2\pi}}, \ldots \tag{8}$$

To show that this is an orthonormal system we just compute the inner product:

$$\begin{aligned}(\phi_\nu, \overline{\phi_\mu}) =& \int_0^{2\pi} \frac{1}{2\pi} e^{i\nu x} \frac{1}{2\pi} e^{-i\mu x} \, dx \\ =& \frac{1}{2\pi} \int_0^{2\pi} e^{i(\nu-\mu)x} \, dx \\ =& \delta_{\nu\,\mu}\end{aligned} \tag{9}$$

since when $\nu = \mu$ the integral is just $\frac{2\pi}{2\pi} = 1$ and when $\nu \neq \mu$ we have that $\cos(2\pi(\nu - \mu) = 1$ and of course $\sin 2\pi(\nu - \mu) = 0$ so that the integral evaluates to zero.

A set of functions $\{f_i\}$ are linearly dependent if $\sum_{i=1}^n c_i f_i = 0$ with constant coefficients c_i not all equal to zero for all x. Otherwise the system is said to be linearly independent. Orthonormal system are always linearly independent, for suppose that $\sum_{i=1}^n c_i \phi_i = 0$ holds, then for any ν:

$$(\phi_\nu, \sum_{i=1}^n c_i \phi_i) = c_\nu \underbrace{(\phi_\nu, \phi_\nu)}_{=1} = 0 \tag{10}$$

Thus $c_\nu = 0$, and since this holds for any ν, the system is linearly independent.

3 Mean square concept

The concept of mean square approximation is based on the fundamental fact that:

$$\int (f - \sum_{\nu=1}^{n} c_\nu \phi_\nu)^2 \, dx \geq 0 \tag{11}$$

We could choose to use any even power for this purpose but 2 will do. The integral of a non-negative quantity must also be non-negative so expanding we have:

$$\begin{aligned}
0 \leq & \int (f - \sum_{\nu=1}^{n} c_\nu \phi_\nu)^2 \, dx \\
0 \leq & \int f^2 \, dx - 2 \sum_{\nu=1}^{n} c_\nu \int f \phi_\nu \, dx + \int \Big(\sum_{\nu=1}^{n} c_\nu \phi_\nu \Big)^2 \, dx \\
= & \int f^2 \, dx - 2 \sum_{\nu=1}^{n} c_\nu \underbrace{\int f \phi_\nu \, dx}_{=c_\nu} + \sum_{\nu=1}^{n} c_\nu^2 \\
0 \leq & \|f\| - 2 \sum_{\nu=1}^{n} c_\nu^2 + \sum_{\nu=1}^{n} c_\nu^2 \\
0 \leq & \|f\| - \sum_{\nu=1}^{n} c_\nu^2 \\
\therefore & \sum_{\nu=1}^{n} c_\nu^2 \leq \|f\|
\end{aligned} \tag{12}$$

Note that in $\int \Big(\sum_{\nu=1}^{n} c_\nu \phi_\nu \Big)^2 \, dx$ the integrals of the cross terms $\int c_\nu c_\mu \phi_\nu \phi_\mu \, dx = 0$ for $\nu \neq \mu$ due to the orthogonality condition.

Since $\|f\|$ is independent of n we can assert that:

$$\boxed{\sum_{\nu=1}^{\infty} c_\nu^2 \leq \|f\|} \tag{13}$$

This is Bessel's inequality and it is true for all orthonormal systems. The sum of the squares of the expansion coefficients always converges. Because the series converges it follows that the constant coefficient terms c_ν converge to zero ie $\lim_{\nu \to \infty} c_\nu = 0$. In the context of Fourier theory this is a well known phenomenon - the Fourier coefficients c_ν always approach zero as $\nu \to \infty$. Thus we can write as a general proposition:

$$\lim_{\nu\to\infty} c_\nu = \lim_{\nu\to\infty} (f, \phi_\nu) = 0 \tag{14}$$

If we denote the partial sums of our linear combination of orthonormal functions by $S_n(x)$:

$$S_n(x) = \sum_{\nu=1}^{n} c_\nu \phi_\nu \tag{15}$$

we say that this sum $S_n(x)$ converges in the mean to the function f , if over the relevant interval:

$$\lim_{n\to\infty} \int [f(x) - S_n(x)]^2 \, dx = 0 \tag{16}$$

This is sometimes written as:

$$\underset{n\to\infty}{\text{l.i.m}}\, S_n(x) = 0 \tag{17}$$

For complex valued functions the inequality corresponding to (12) holds with the inner product and coefficients given as follows:

$$\sum_{\nu=1}^{n} c_\nu^2 \leq \|f\| = (f, \bar{f}) \tag{18}$$

where $c_\nu = (f, \bar{\phi_\nu})$

The proof of the inequality follows similar lines taking account of complex conjugates in the inner product:

$$
\begin{aligned}
0 \leq \int |f - \sum_{\nu=1}^{n} c_\nu \phi_\nu|^2 \, dx =& \int \Big(f - \sum_{\nu=1}^{n} c_\nu \phi_\nu) \overline{\Big(f - \sum_{\nu=1}^{n} c_\nu \phi_\nu\Big)} \, dx \\
=& \int f \bar{f} \, dx - \int f \sum_{\nu=1}^{n} \overline{c_\nu} \overline{\phi_\nu} \, dx - \int \bar{f} \sum_{\nu=1}^{n} c_\nu \phi_\nu \, dx + \int \Big(\sum_{\nu=1}^{n} c_\nu \phi_\nu\Big) \Big(\sum_{\nu=1}^{n} \overline{c_\nu} \overline{\phi_\nu}\Big) \, dx \\
=& \int |f|^2 - \sum_{\nu=1}^{n} \overline{c_\nu} \underbrace{\int f \bar{\phi_\nu} \, dx}_{=c_\nu} - \sum_{\nu=1}^{n} c_\nu \underbrace{\int \bar{f} \phi_\nu \, dx}_{=\overline{c_\nu}} \, dx + \sum_{\nu=1}^{n} |c_\nu|^2 \\
=& \|f\| - 2 \sum_{\nu=1}^{n} |c_\nu|^2 + \sum_{\nu=1}^{n} |c_\nu|^2 \\
=& \|f\| - \sum_{\nu=1}^{n} |c_\nu|^2 \\
\therefore& \|f\| \geq \sum_{\nu=1}^{n} |c_\nu|^2
\end{aligned}
\tag{19}
$$

4 Approximating a function by a linear combination of basis functions

One way, the most basic, of approximating a continuous function is by a linear combination of the building blocks or basis functions ie $\sum_{\nu=1}^{n} \lambda_\nu \phi_\nu$ where the coefficients λ_ν are constant and n is fixed so that the mean square error M is as small as possible where:

$$
M = \int \Big(f - \sum_{\nu=1}^{m} \lambda_\nu \phi_\nu\Big)^2 \, dx \tag{20}
$$

Now

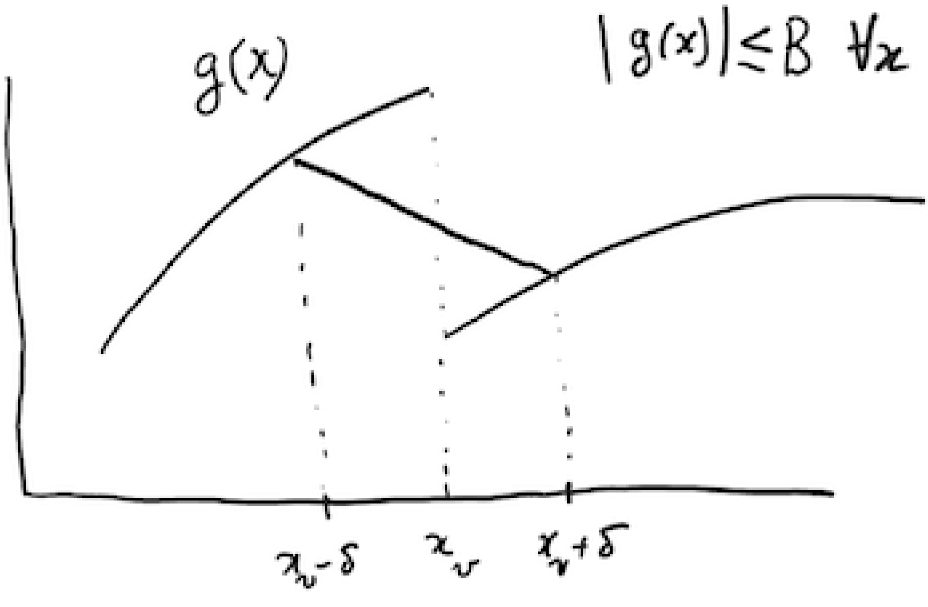
$$
\begin{aligned}
M =& \int f^2 \, dx - 2 \int f \sum_{\nu=1}^{n} \lambda_\nu \phi_\nu \, dx + \int \Big(\sum_{\nu=1}^{n} \lambda_\nu \phi_\nu\Big)^2 \, dx \\
=& \int f^2 \, dx - 2 \sum_{\nu=1}^{n} \lambda_\nu (f, \phi_\nu) \int + \sum_{\nu=1}^{n} \lambda_\nu^2 \\
=& \int f^2 \, dx - 2 \sum_{\nu=1}^{n} \lambda_\nu c_\nu + \sum_{\nu=1}^{n} \lambda_\nu^2 \\
=& \int f^2 \, dx + \sum_{\nu=1}^{n} (\lambda_\nu - c_\nu)^2 - \sum_{\nu=1}^{n} c_\nu^2
\end{aligned}
\tag{21}
$$

and so M is minimised when $\lambda_\nu = c_\nu$. ie the coefficients of f are simply the expansion coefficients in the orthonormal basis. Completeness of the system of orthonormal functions ϕ_ν means that you can make the mean square error $\int (f - \sum_{\nu=1}^{n} c_\nu \phi_\nu)^2 \, dx$ less than any arbitrary $\epsilon > 0$. For complete orthonormal systems Bessel's inequality becomes an equality:

$$\sum_{\nu=1}^{\infty} c_\nu^2 = \|f\| \tag{22}$$

If this equation is satisfied for all continuous functions f then the system of functions ϕ_ν is complete.

Any piecewise continuous function g may be approximated by a continuous function f in such a way that $\int (f-g)^2 \, dx$ is arbitrarily small. If we let the points of discontinuity be x_ν we can approximate f in each sub-interval $[x_\nu - \delta, x_\nu + \delta]$, where $\delta > 0$, by replacing the graph of g by the straight line between $(x_\nu - \delta, g(x_\nu - \delta))$ and $(x_\nu + \delta, g(x_\nu + \delta))$ as indicated in the diagram below.



Note that $f(x) = g(x)$ for points outside $[x_\nu - \delta, x_\nu + \delta]$ so the mean square error for those points is zero. To show that in the interval $[x_\nu - \delta, x_\nu + \delta]$ we can make the mean square error arbitrarily small we note that $(f(x) - g(x))^2$ is continuous by construction on $[x_\nu - \delta, x_\nu + \delta]$ and by the Mean Value Theorem there exists a $y \in (x_\nu - \delta, x_\nu + \delta)$ such that:

$$\begin{aligned}\int_{x_\nu-\delta}^{x_\nu+\delta}(f(x)-g(x))^2\,dx =&2\delta[f(y)-g(y)]^2\\ \leq&2\delta[|f(y)|+|g(y)|]^2\\ \leq 2\delta(2B)^2 = 8\delta B^2\end{aligned} \tag{23}$$

If there are k discontinuities then by the linearity of the integration process and replication of the above reasoning we can see that we can make $\int(f(x)-g(x))^2\,dx \leq 8k\delta B^2$ over the k discontinuities and clearly we can make this arbitrarily small by making δ as small as needed. Because of the continuity this is possible.

For n sufficiently large we can make the mean square error integral $\int(f-\sum_{\nu=1}^{n} c_\nu\phi_\nu)^2\,dx$ arbitrary small where the expansion coefficients $c_1, c_2, \ldots$ of f will be arbitrarily small. Given that we then have the following:

$$\begin{aligned}M' =& \int\Big(g-\sum_{\nu=1}^{n}c_\nu\phi_\nu\Big)^2\,dx\\ =&\int\Big[\Big(g-f\Big)+\Big(f-\sum_{\nu=1}^{n}c_\nu\phi_\nu\Big)\Big]^2\,dx\\ =&\|g-f\|+\|f-\sum_{\nu=1}^{n}c_\nu\phi_\nu\|+2\Big(g-f,f-\sum_{\nu=1}^{n}c_\nu\phi_\nu\Big)\\ \leq&\|g-f\|+\underbrace{\|f-\sum_{\nu=1}^{n}c_\nu\phi_\nu\|}_{<\epsilon}+2\sqrt{\|g-f\|\cdot\underbrace{\|f-\sum_{\nu=1}^{n}c_\nu\phi_\nu\|}_{<\epsilon}}\end{aligned} \tag{24}$$

Note that we have used the Cauchy-Schwarz inequality above in the form of $\big(2(p,q)\big)^2 \leq 2(p,p)\,2(q,q) = 4\|p\|\|q\|$. Hence $2(p,q) \leq 2\sqrt{\|p\|\|q\|}$.

Because we can make $\int(f-\sum_{\nu=1}^{n}c_\nu\phi_\nu)^2\,dx$ arbitrarily small we can make M' arbitrarily close to $\|g-f\|$ and if the expansion coefficients of g in the orthonormal basis are a_ν then because the a_ν give the least mean square error for g we have the following:

$$M = \int(g-\sum_{\nu=1}^{n}a_\nu\phi_\nu)^2\,dx \leq M' \tag{25}$$

Thus the completeness relationship holds for g if it holds for f.

Because the completeness of an orthonormal set ϕ_ν can be expressed as:

$$\lim_{n\to\infty}\int (f - \sum_{\nu=1}^{n} c_\nu\phi_\nu)^2\,dx = 0 \tag{26}$$

it is tempting to conclude that $f = \sum_{\nu=1}^{\infty} c_\nu\phi_\nu$ (ie f an be expanded in a series of the functions ϕ_ν, but this is only valid if the series $\sum_{\nu=1}^{\infty} c_\nu\phi_\nu$ is uniformly convergent. Uniform convergence is needed to push the limiting process through the integral.

A continuous piecewise function is uniquely determined by its expansion coefficients with respect to a given orthonormal system. For suppose that $f = \sum_{\nu=1}^{n} c_\nu\phi_\nu$ and also that $g = \sum_{\nu=1}^{n} c_\nu\phi_\nu$. Then $\|f-g\| = 0$ because of the equality of coefficients. This uniqueness for complete orthonormal system occurs even though there is only convergence in the mean.

5 Relationship between pointwise and mean square convergence

A series may converge in the mean without converging at each point. Even if an infinite series converges at every point, it may not converge in the mean. Indeed, mean square convergence may even become unbounded. This can be summarised as:

Pointwise convergence of $S_n(x)$ does not imply mean square convergence and conversely.

This can be appreciated from the following examples.

5.1 Example 1

Suppose that:

$$S_n(x) = n\sqrt{x}\,e^{\frac{-nx^2}{2}} \text{ for } 0 \le x \le 1 \tag{27}$$

Now if $x = 0$ then $S_n(x) = 0$ for all n and if $0 < x \le 1$ we can let $u = \sqrt{\frac{n}{2}}x$. Then for any fixed x, $u \to \infty$ as $n \to \infty$ and $u^2 = \frac{n}{2}x^2$. Thus for any fixed x in $0 < x \le 1$, $S_n(x) = S_n\left(\sqrt{\frac{2}{n}}\right)x = \sqrt{2}ue^{-u^2}$ which clearly converges to 0 as u and hence n, approaches ∞ since the exponential dominates any polynomial power. Thus we have pointwise convergence ie $\lim_{n\to\infty} S_n(x) = 0$.

For mean square convergence we have (noting that our function is pointwise null):

$$
\begin{aligned}
M_n &= \int_0^1 [0 - S_n(x)]^2 \, dx \\
&= \int_0^1 [-n\sqrt{x} e^{\frac{-nx^2}{2}}]^2 \, dx \\
&= \int_0^1 n^2 x e^{-nx^2} \, dx \\
&= \int_0^n \frac{n}{2} e^{-u} \, du \text{ with the substitution } u = nx^2 \\
&= \frac{n}{2}\Big[-e^{-u}\Big]_0^n \\
&= \frac{n}{2}(1 - e^{-n}) \to \infty \text{ as } n \to \infty
\end{aligned} \tag{28}
$$

Thus although we have pointwise convergence at every point in $[0, 1]$ we do not have mean convergence.

5.2 Example 2

Let $S_n(x) = x^n$ for $0 \leq x \leq 1$ and let $f(x) = 0$ for $0 \leq x \leq 1$. The mean square convergence for this setup is:

$$
\begin{aligned}
M_n &= \int_0^1 [f(x) - S_n(x)]^2 \, dx \\
&= \int_0^1 [-x^n]^2 \, dx \\
&= \int_0^1 x^{2n} \, dx \\
&= \Big[\frac{x^{2n+1}}{2n+1}\Big]_0^1 \\
&= \frac{1}{2n+1} \to 0 \text{ as } n \to \infty
\end{aligned} \tag{29}
$$

Thus we have mean square convergence. However, for $0 \leq x < 1$ $x^n \to 0$ as $n \to \infty$ but when $x = 1$, $x^n = 1$ for all n and so there is not pointwise convergence.
For more on mean square convergence see [2], pages 525-533.

6 References

[1] R Courant and D Hilbert “Methods of Mathematical Physics Volume 1”, Wiley Classics Edition, 1989.

[2] William E Boyce and Richard C DiPrima "Elementary Differential Equations and Boundary Value Problems" , Second Edition, Wiley, 1969.

7 History

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