Singular Value Decomposition worked numerical examples

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1 Introduction

This article is limited to the numerical aspects of Singular Value Decomposition (SVD) rather than the detailed linear algebraic proofs that underlie the theory. It is in a tutorial format rather than just presenting a solution. For more detail on the theory I can recommend Gil Strang's MIT OpenCourseWare course 18.065: `https://ocw.mit.edu/courses/mathematics/18-065-matrix-methods-in-data-analysis-signal-processing-and-machine-learning-spring-2018/`. For some reason pdfLatex does not want this link to work despite many efforts, so you will have to copy it.

Gil is in his 80s and brings a lifetime of knowledge to this area which he has updated for deep learning applications.

Relevant theory is to be had in Chapter 7 of Gil Strang's textbook for MIT course 18.065: `http://math.mit.edu/classes/18.095/2016IAP/lec2/SVD_Notes.pdf` In what follows everything lives in real space rather than complex space. Thus all references are to orthogonal matrices rather than unitary matrices. This does not prejudice the overall generality of the approach set out below.

2 Example 1

We start with a 2 x 2 matrix with real entries:

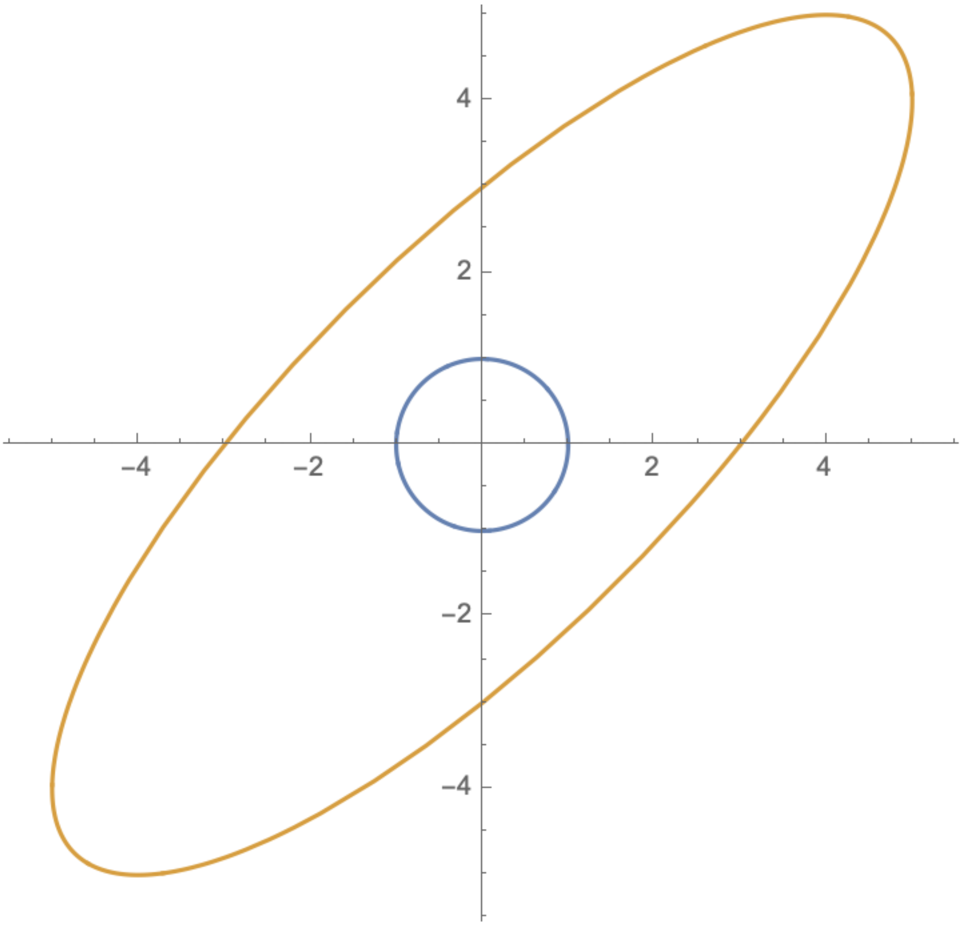
$$A = \begin{pmatrix} 3 & 4 \\ 0 & 5 \end{pmatrix} \tag{1}$$

Note here that SVD theory deals with the factorisation of general m x n matrices rather than just square matrices.

SVD posits the existence of 2 rotation matrices and one stretching matrix which form the factorisation of A thus:

$$A = U\Sigma V^T \tag{2}$$

What this factorisation says is that, you give me a matrix A and I can break it down into rotations and stretches. Thus the matrix A in (1) turns the unit circle into an ellipse as shown below:



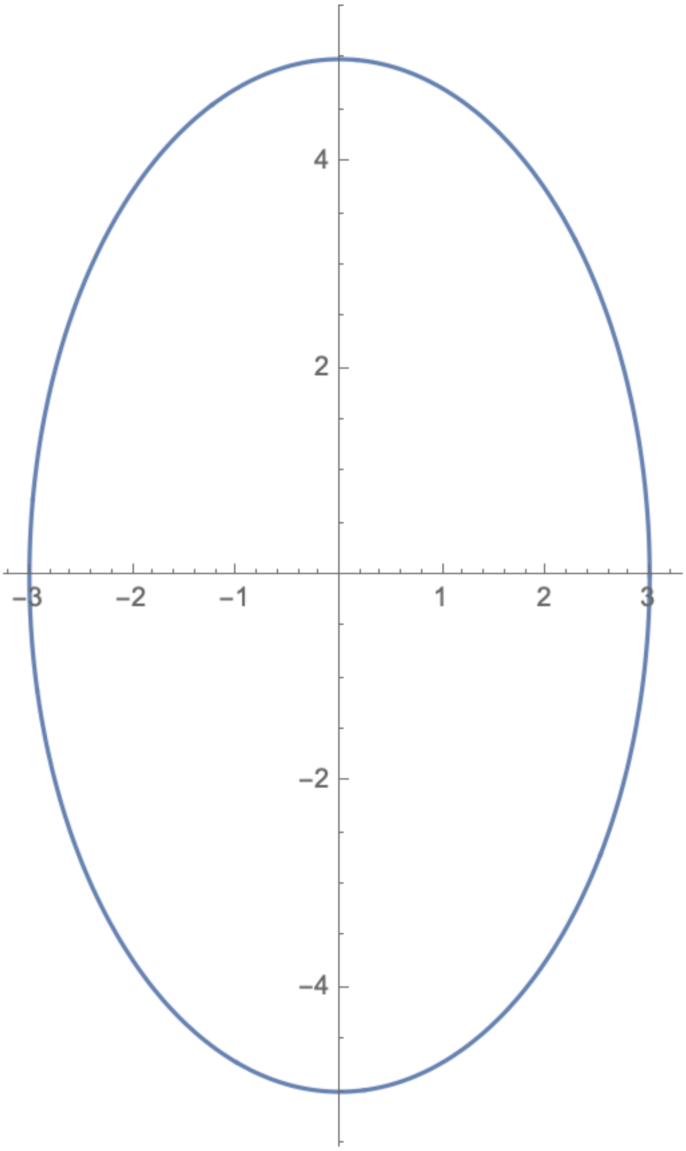
If you are wondering how to generate the above graph, just recall that the unit circle can be parametrically defined as:

$$f(\theta) = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \tag{3}$$

for $0 \leq \theta \leq 2\pi$, so all we have to do is use ParametricPlot in Mathematica or the equivalent in MATLAB as follows: ParametricPlot$[\{f[\theta], f[\theta].A\}, \{\theta, 0, 2\pi\}]$. You should ask yourself how you prove this analytically without recourse to the crutch of Mathematica or MATLAB. Just do the matrix product:

$$\begin{pmatrix} 3 & 4 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = \begin{pmatrix} 3\cos\theta + 4\sin\theta \\ 5\sin\theta \end{pmatrix} = \begin{pmatrix} 3\cos\theta \\ 5\sin\theta \end{pmatrix} + \begin{pmatrix} 4\sin\theta \\ 0 \end{pmatrix} \tag{4}$$

You should recognise $\begin{pmatrix} 3\cos\theta \\ 5\sin\theta \end{pmatrix}$ as the parametric form for an ellipse which looks like this:



The remaining term $\begin{pmatrix} 4\sin\theta \\ 0 \end{pmatrix}$ has the effect of rotating the ellipse and this can be confirmed by adding the two components. Equally $\begin{pmatrix} 3\sin\theta \\ 5\cos\theta \end{pmatrix}$ will give you an ellipse. Why? More fundamentally we can write (4) as:

$$\begin{pmatrix} 3\cos\theta + 4\sin\theta \\ 5\sin\theta \end{pmatrix} = \begin{pmatrix} 5\cos\left[\theta - \arctan\left(\frac{4}{3}\right)\right] \\ 5\sin\theta \end{pmatrix} \tag{5}$$

which has the same form as the standard parametric form for an ellipse: $\begin{pmatrix} a\cos\theta \\ b\sin\theta \end{pmatrix}$. Can you see how to work (5) out? Hint: Expand $a\cos(\theta - \gamma)$ and do some equating of coefficients.

There are other ways of showing that you get an ellipse but these are simple ways to establish the point.

In (2) U is the left orthogonal matrix.

Σ is the matrix of singular values in decreasing order of magnitude (this is a convention).

V is the right orthogonal matrix.

The recipe for finding the SVD of any matrix is to start with the left hand and right hand symmetric forms:

(1) AA^T– this correlates with the left orthogonal matrix U
(2) A^TA – this correlates with the right orthogonal matrix V^T.

A^T means the transpose of A.

(3) We then find the eigenvalues and eigenvectors of AA^T and A^TA
(4) As part of the process in (3) we establish a matrix of the singular values (σ_i) of A. Recall that:

$$\begin{aligned} AA^T =& \left(U\Sigma V^T\right)\left(U\Sigma V^T\right)^T \\ =& U\Sigma V^T V\Sigma^T U^T \\ =& U\left(\Sigma\Sigma^T\right)U^T \end{aligned} \tag{6}$$

If λ_i is an eigenvalue of AA^T then the last line of (5) explains why we get the basic relationship:

$$\lambda_i = \sigma_i^2 \tag{7}$$

since the term $\Sigma\Sigma^T$ will give rise to terms σ_i^2 because it has a diagonal structure (and note that $\Sigma\Sigma^T$ is square even if Σ is not).

In this example:

$$\begin{aligned} AA^T &= \begin{pmatrix} 3 & 4 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 4 & 5 \end{pmatrix} \\ &= \begin{pmatrix} 25 & 20 \\ 20 & 25 \end{pmatrix} \end{aligned} \tag{8}$$

Note that AA^T is symmetric and this is the case in general for any m x n matrix not just square matrices. Thus:

$$\left(AA^T\right)^T = \left(A^T\right)^T A^T = AA^T \tag{9}$$

If A is m x n then AA^T is m x m.

The eigenvectors of AA^T solve this equation:

$$(AA^T - \lambda I)\vec{u} = \vec{0} \tag{10}$$

and to find the eigenvalues λ_i we simply solve:

$$\det(AA^T - \lambda I) = 0 \tag{11}$$

The 2 x 2 case as in this example is simple because we know that (10) can be written as:

$$\lambda^2 - \operatorname{tr}(AA^T)\lambda + \det(AA^T) = 0 \tag{12}$$

where "tr" is the trace of a matrix ie sum of diagonal elements. Thus from (8) we have:

$$\lambda^2 - 50\lambda + 225 = (\lambda - 45)(\lambda - 5) = 0 \tag{13}$$

giving the two eigenvalues in descending order as $\lambda_1 = 45$ and $\lambda_2 = 5$.

One can also write out the determinant in (11) as follows and proceed from there:

$$\begin{vmatrix} 25 - \lambda & 20 \\ 20 & 25 - \lambda \end{vmatrix} = 0 \tag{14}$$

The corresponding unit eigenvectors (which will be orthogonal and hence orthonormal) can be found by inspection:

For $\lambda_1 = 45$ we have:

$$\begin{pmatrix} -20 & 20 \\ 20 & -20 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \vec{0} \tag{15}$$

hence a unit vector is $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$.

For $\lambda_1 = 5$ we have:

$$\begin{pmatrix} 20 & 20 \\ 20 & 20 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \vec{0} \tag{16}$$

hence a unit vector is $\begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$. Hence our left orthogonal matrix is:

$$U = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = [\vec{u_1}, \vec{u_2}] \tag{17}$$

Note that $\vec{u_1}$ and $\vec{u_2}$ are orthogonal (do the dot product). Note that in (16) we could have chosen an equally legitimate orthonormal eigenvector for $\vec{u_2}$:

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix} \tag{18}$$

I will return to this later, but you should think how that choice of eigenvector might play out.

To find the right singular eigenvectors we need to find the eigenvectors which solve the following equation. We know that the eigenvalues will be the same (as can be proved in general - for any real symmetric matrix A, its eigenvalues are the same as that of A^T) but for the sake of demonstration we can check that as we go:

$$\begin{aligned} A^T A &= \begin{pmatrix} 3 & 0 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} 3 & 4 \\ 0 & 5 \end{pmatrix} \\ &= \begin{pmatrix} 9 & 12 \\ 12 & 41 \end{pmatrix} \end{aligned} \tag{19}$$

Hence:

$$A^T A - \lambda I = \begin{pmatrix} 9 - \lambda & 12 \\ 12 & 41 - \lambda \end{pmatrix} \tag{20}$$

Quickly confirming the eigenvalues are the same:

$$\lambda^2 - \text{tr}(A^TA) + \det(A^TA) = \lambda^2 - 50\lambda + (369 - 144) = \lambda^2 - 50\lambda + 255 = 0 \quad (21)$$

as before.

For $\lambda_1 = 45$ we have:

$$\begin{pmatrix} -36 & 12 \\ 12 & -4 \end{pmatrix} \vec{v_1} = \vec{0} \quad (22)$$

By inspection:

$$\vec{v_1} = \begin{pmatrix} \frac{1}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} \end{pmatrix} \quad (23)$$

For $\lambda_2 = 5$ we have:

$$\begin{pmatrix} 4 & 12 \\ 12 & 36 \end{pmatrix} \vec{v_2} = \vec{0} \quad (24)$$

By inspection:

$$\vec{v_2} = \begin{pmatrix} \frac{-3}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \end{pmatrix} \quad (25)$$

Therefore:

$$V = \begin{pmatrix} \frac{1}{\sqrt{10}} & \frac{-3}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \end{pmatrix} \quad (26)$$

We know from general theory that for orthgonal matrices $V^{-1} = V^T$ and you can mentally check that this is the case with (26). If we were dealing with unitary matrices over the complex field the relationship would be $V = V^*$ where V^* is the complex conjugate transpose of V.

Note that from (2), we must have (and this is true in general):

$$AV = U\Sigma \quad (27)$$

(27) is justified because V is an orthogonal matrix so $V^{-1} = V^T$ and we postmultiply both sides of (2) by V^{-1} and then (2) follows.

Here we have:

$$\Sigma = \begin{pmatrix} \sqrt{45} & 0 \\ 0 & \sqrt{5} \end{pmatrix} = \begin{pmatrix} 3\sqrt{5} & 0 \\ 0 & \sqrt{5} \end{pmatrix} \quad (28)$$

Note that the singular values in the matrix Σ are the square roots of the λs ie $\sigma_i = \sqrt{\lambda_i}$:

Using (17), (26) and (27) we have:

$$\begin{aligned}
AV &= \begin{pmatrix} 3 & 4 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{10}} & \frac{-3}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \end{pmatrix} \\
&= \begin{pmatrix} \frac{15}{\sqrt{10}} & \frac{-5}{\sqrt{10}} \\ \frac{15}{\sqrt{10}} & \frac{5}{\sqrt{10}} \end{pmatrix} \\
&= \begin{pmatrix} \frac{3\sqrt{5}}{\sqrt{2}} & \frac{-\sqrt{5}}{\sqrt{2}} \\ \frac{3\sqrt{5}}{\sqrt{2}} & \frac{\sqrt{5}}{\sqrt{2}} \end{pmatrix} \\
U\Sigma &= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 3\sqrt{5} & 0 \\ 0 & \sqrt{5} \end{pmatrix} \\
&= \begin{pmatrix} \frac{3\sqrt{5}}{\sqrt{2}} & \frac{-\sqrt{5}}{\sqrt{2}} \\ \frac{3\sqrt{5}}{\sqrt{2}} & \frac{\sqrt{5}}{\sqrt{2}} \end{pmatrix}
\end{aligned} \tag{29}$$

Thus our left and right eigenmatrices work and we can be confident that $A = U\Sigma V^T$ which is confirmed below:

$$\begin{aligned}
U\Sigma V^T &= \begin{pmatrix} \frac{3\sqrt{5}}{\sqrt{2}} & -\frac{\sqrt{5}}{\sqrt{2}} \\ \frac{3\sqrt{5}}{\sqrt{2}} & \frac{\sqrt{5}}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \\ \frac{-3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \end{pmatrix} \\
&= \begin{pmatrix} 3 & 4 \\ 0 & 5 \end{pmatrix} \\
&= A
\end{aligned} \tag{30}$$

Could we work out V from the first line of (29)? In this case because A is invertible we could simply premultiply both sides of (29) by A^{-1} and this will gives us V, but in general A is not a square matrix so this approach does not generalise.

For SVD to work we must have:

$$\begin{aligned}
A\vec{v_1} &= \sigma_1 \vec{u_1} \\
A\vec{v_2} &= \sigma_2 \vec{u_2}
\end{aligned} \tag{31}$$

Checking:

$$
\begin{aligned}
A\vec{v_1} =& \begin{pmatrix} 3 & 4 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} \end{pmatrix} \\
=& \begin{pmatrix} \frac{3\sqrt{5}}{\sqrt{2}} \\ \frac{3\sqrt{5}}{\sqrt{2}} \end{pmatrix} \\
=& 3\sqrt{5} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \\
=& \sigma_1 \vec{u_1}
\end{aligned} \tag{32}
$$

$$
\begin{aligned}
A\vec{v_2} =& \begin{pmatrix} 3 & 4 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} \frac{-3}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \end{pmatrix} \\
=& \begin{pmatrix} \frac{-5}{\sqrt{10}} \\ \frac{5}{\sqrt{10}} \end{pmatrix} \\
=& \sqrt{5} \begin{pmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \\
=& \sigma_2 \vec{u_2}
\end{aligned} \tag{33}
$$

Going back to the discussion around (17) about the choice of $\vec{u_2}$ what would we get if we had chosen it? Let's just see if (32) holds with the alternative eigenvector $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix}$:

$$
\sigma_2 \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix} = -\sigma_2 \vec{u_2} \neq A\vec{v_2} \tag{34}
$$

We are out by a factor of -1 so be aware that this can happen. There is uniqueness up to sign. However, if we did choose $\vec{u_2} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix}$ then we can still get consistent results if we choose $\vec{v_2} = \begin{pmatrix} \frac{3}{\sqrt{10}} \\ \frac{-1}{\sqrt{10}} \end{pmatrix}$ rather than $\begin{pmatrix} \frac{-3}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \end{pmatrix}$ (ie multiply original $\vec{v_2}$ by -1) since:

$$
A\vec{v_2} = \begin{pmatrix} 3 & 4 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} \frac{3}{\sqrt{10}} \\ \frac{-1}{\sqrt{10}} \end{pmatrix} = \begin{pmatrix} \frac{5}{\sqrt{10}} \\ \frac{-5}{\sqrt{10}} \end{pmatrix} = \sqrt{5} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix} \tag{35}
$$

2.1 Geometrical interpretation

Recall that from (2) the action of A on a vector $\vec{x}$ is a rotation, (V^T), followed by a stretch (Σ)then a final rotation (U). The rotations do not change lengths. To see this in detail take:

$$\vec{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{36}$$

The action of A on $\vec{x}$ gives:

$$A\vec{x} = \begin{pmatrix} 3 & 4 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix} \tag{37}$$

We now go the decomposition route:

$$V^T\vec{x} = \begin{pmatrix} \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \\ \frac{-3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{4}{\sqrt{10}} \\ \frac{-2}{\sqrt{10}} \end{pmatrix} \tag{38}$$

Note $\|\vec{x}\| = \sqrt{2}$ and $\|V^T\vec{x}\| = \sqrt{\frac{16}{10} + \frac{4}{10}} = \sqrt{2}$. So length is indeed preserved.

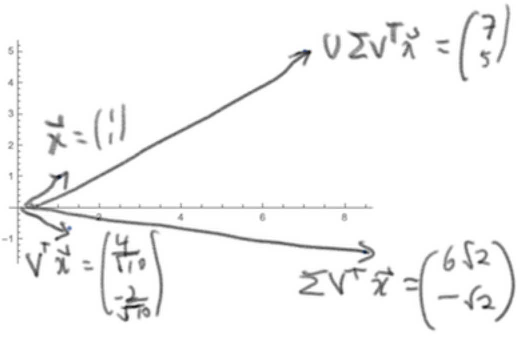
$$\Sigma V^T\vec{x} = \begin{pmatrix} 3\sqrt{5} & 0 \\ 0 & \sqrt{5} \end{pmatrix} \begin{pmatrix} \frac{4}{\sqrt{10}} \\ \frac{-2}{\sqrt{10}} \end{pmatrix} = \begin{pmatrix} 6\sqrt{2} \\ -\sqrt{2} \end{pmatrix} \tag{39}$$

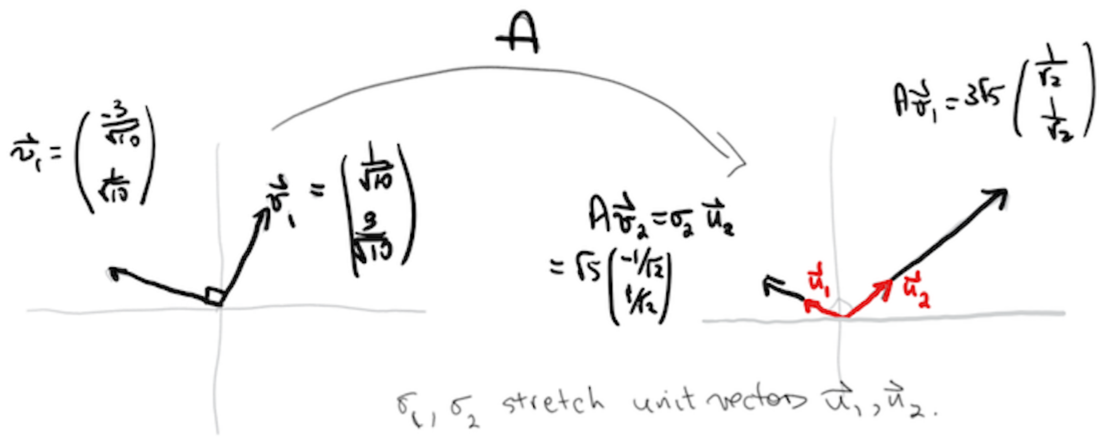
Note $\|\Sigma V^T\vec{x}\| = \sqrt{74}$.

Finally we get:

$$U\Sigma V^T\vec{x} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 6\sqrt{2} \\ -\sqrt{2} \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix} \tag{40}$$

Once again length is preserved: $\|U\Sigma V^T\vec{x}\| = \sqrt{74}$.





3 Example 2

This time we work with a rectangular 2 x 3 matrix:

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \tag{41}$$

We start by calculating AA^T and A^TA:

$$AA^T = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \tag{42}$$

$$A^TA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \tag{43}$$

For the left eigenmatrix we need to solve:

$$\det(AA^T - \lambda I) = \begin{vmatrix} \lambda - 2 & 0 \\ 0 & \lambda - 1 \end{vmatrix} = (\lambda - 2)(\lambda - 1) = 0 \tag{44}$$

which gives $\lambda_1 = 2$ and $\lambda_2 = 1$.

For $\lambda_1 = 2$ the eigenvector can be found by inspection:

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{45}$$

Hence $\vec{u_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Similarly for $\lambda_2 = 1$:

$$\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{46}$$

Hence $\vec{u_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Note that $\vec{u_1}$ and $\vec{u_2}$ are orthogonal unit vectors. Hence our left eigenmatrix corresponding to the eigenvalues in decreasing size:

$$U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I \tag{47}$$

At this stage we can note that we only have 2 eigenvalues but our Σ must be a 2 x 3 matrix (since V^T is a 3 x 3 matrix) , suggesting a third eigenvalue which indeed turns out to be the case. For the right eigenmatrix V we must solve:

$$\begin{aligned} \det(A^T A - \lambda I) = & \begin{vmatrix} 1-\lambda & 0 & -1 \\ 0 & 1-\lambda & 0 \\ -1 & 0 & 1-\lambda \end{vmatrix} \\ =& (1-\lambda)(1-\lambda)^2 - (1-\lambda) \\ =& (1-\lambda)(\lambda^2 - 2\lambda) \\ =& \lambda(\lambda-2)(\lambda-1) \\ =& 0 \end{aligned} \tag{48}$$

Thus our eigenvalues in decreasing order are: $\lambda_1 = 2, \lambda_2 = 1, \lambda_3 = 0$ and thus our matrix of singular values $\sigma_i = \sqrt{\lambda_i}$ becomes :

$$\Sigma = \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{49}$$

For $\lambda_1 = 2$ our eigenvector can be found by inspection:

$$\begin{pmatrix} -1 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{50}$$

Hence our unit vector is:

$$\vec{v_1} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{-1}{\sqrt{2}} \end{pmatrix} \tag{51}$$

For $\lambda_2 = 1$ we have:

$$\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{52}$$

Hence our unit vector is:

$$\vec{v_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \tag{53}$$

Note that $\vec{v_1}$ and $\vec{v_2}$ are orthogonal.

Finally for $\lambda_3 = 0$ we have:

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{54}$$

Hence our unit vector is:

$$\vec{v_3} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} \tag{55}$$

Note that $\vec{v_3}$ is orthogonal to $\vec{v_1}$ and $\vec{v_2}$.

Thus we have :

$$V = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \tag{56}$$

and

$$V^T = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \tag{57}$$

We now confirm that the decomposition gives us A (noting (47)):

$$\begin{aligned} U\Sigma V^T =& I\Sigma V^T \\ =& \Sigma V^T \\ =& \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \\ =& \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \\ =& A \end{aligned} \tag{58}$$

We must have $A\vec{v_i} = \sigma_i \vec{u_i}$ for $i = 1, 2, 3$. Let's check it.

$$\begin{aligned} A\vec{v_1} =& \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{-1}{\sqrt{2}} \end{pmatrix} \\ =& \begin{pmatrix} \sqrt{2} \\ 0 \end{pmatrix} \\ =& \sqrt{2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ =& \sigma_1 \vec{u_1} \end{aligned} \tag{59}$$

$$\begin{aligned} A\vec{v_2} =& \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \\ =& \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ =& 1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ =& \sigma_2 \vec{u_2} \end{aligned} \tag{60}$$

$$\begin{aligned} A\vec{v_3} =& \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} \\ =& \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned} \tag{61}$$

But where is $\vec{u_3}$? It does not exist of course as a separate orthonormal eigenvector of AA^T since its null space has dimension 2 with only 2 independent basis vectors and the way to make sense of the final line of (61) is that:

$$A\vec{v_3} = 0(c_1\vec{u_1} + c_2\vec{u_2}) = \sigma_3\vec{u_3} \tag{62}$$

where $\vec{u_3} = c_1\vec{u_1}+c_2\vec{u_2}$ and c_1, c_2 are arbitrary constants. ie $\vec{u_3}$ is a linear combination of the other two vectors which will of course satisfy $A\vec{v_3} = \sigma_3\vec{u_3}$ since $\sigma_3 = 0$.

4 Example 3

Let:

$$A = \begin{pmatrix} 1 & -2 & 0 \\ 0 & -2 & 1 \end{pmatrix} \tag{63}$$

We first find AA^T and A^TA.

$$\begin{aligned} AA^T &= \begin{pmatrix} 1 & -2 & 0 \\ 0 & -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -2 & -2 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix} \end{aligned} \tag{64}$$

$$\begin{aligned} A^TA &= \begin{pmatrix} 1 & 0 \\ -2 & -2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -2 & 0 \\ 0 & -2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & -2 & 0 \\ -2 & 8 & -2 \\ 0 & -2 & 1 \end{pmatrix} \end{aligned} \tag{65}$$

Working out the eigenvalues of the left eigenmatrix U we have:

$$\begin{aligned} \det(AA^T - \lambda I) =& \begin{vmatrix} 5-\lambda & 4 \\ 4 & 5-\lambda \end{vmatrix} \\ =& \lambda^2 - 10\lambda + 9 \\ =& (\lambda - 9)(\lambda - 1) \\ =& 0 \end{aligned} \tag{66}$$

Thus $\lambda_1 = 9$ and $\lambda_2 = 1$.

The eigenvector corresponding to λ_1 is found by inspection:

$$\begin{pmatrix} -4 & 4 \\ 4 & -4 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{67}$$

Hence:

$$\vec{u_1} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \tag{68}$$

The eigenvector corresponding to λ_2 also found by inspection:

$$\begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{69}$$

Hence:

$$\vec{u_2} = \begin{pmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \tag{70}$$

So our left eigenmatrix is:

$$U = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \tag{71}$$

What happens if, instead of $\vec{u_1} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$ we choose the equally valid eigenvector $\vec{\mu_1} = \begin{pmatrix} \frac{-1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix}$ (it solves (68)) and we replace $\vec{u_2} = \begin{pmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$ by $\vec{\mu_2} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix}$ (it solves (69)) ? Note that $\vec{\mu_1}$ and $\vec{\mu_2}$ are orthogonal. Our new left eigenmatrix Ω would become:

$$\Omega = \begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix} \tag{72}$$

Go back and read the discussion around (34).

Our next job is to work out the right eigenmatrix V which involves solving a more complicated determinant:

$$
\begin{aligned}
\det(A^TA-\lambda I) = &\begin{vmatrix} 1-\lambda & -2 & 0 \\ -2 & 8-\lambda & -2 \\ 0 & -2 & 1-\lambda \end{vmatrix} \\
=&(1-\lambda)\big[(8-\lambda)(1-\lambda)-4\big]-4(1-\lambda) \\
=&(1-\lambda)^2(8-\lambda)-8(1-\lambda) \\
=&(1-\lambda)(8-9\lambda+\lambda^2-8) \\
=&\lambda(1-\lambda)(\lambda-9) \\
=&0
\end{aligned} \tag{73}
$$

Thus $\lambda_1 = 9, \lambda_2 = 1, \lambda_3 = 0$. Our stretching matrix of singular values ($\sigma_i = \sqrt{\lambda_i}$) is:

$$
\Sigma = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{74}
$$

It is still possible to work out the eigenvalues by inspection (because I cooked this example up so that it could be done that way !). Thus we have for $\lambda_1 = 9$:

$$
\begin{pmatrix} -8 & -2 & 0 \\ -2 & -1 & -2 \\ 0 & -2 & -8 \end{pmatrix} \begin{pmatrix} 1 \\ -4 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{75}
$$

Thus:

$$
\vec{v_1} = \begin{pmatrix} \frac{1}{3\sqrt{2}} \\ \frac{-4}{3\sqrt{2}} \\ \frac{1}{3\sqrt{2}} \end{pmatrix} \tag{76}
$$

since our normalising length is $\sqrt{18} = 3\sqrt{2}$. You should check that $A\vec{v_1} = \sigma_1\vec{u_1}$ (it does).

If you prefer a mechancial row reduction approach the the following sequence gets you the same result: $R_1 \to R_1 - 4R_2, R_3 \to R_3 + R_1, R_1 \to R_1/2, R_2 \to R_2 + R_1, R_2 \to R_2/2$. This gets you to:

$$
\begin{pmatrix} 0 & 1 & 4 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -4 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{77}
$$

The eigenvector corresponding to $\lambda_2 = 1$ is seen to be:

$$\begin{pmatrix} 0 & -2 & 0 \\ -2 & 7 & -2 \\ 0 & -2 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{78}$$

Thus:

$$\vec{v_2} = \begin{pmatrix} \frac{-1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} \tag{79}$$

Again check that $A\vec{v_2} = \sigma_2 \vec{u_2}$.

The eigenvector corresponding to $\lambda_3 = 0$ is seen to be:

$$\begin{pmatrix} 1 & -2 & 0 \\ -2 & 8 & -2 \\ 0 & -2 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{80}$$

Thus:

$$\vec{v_3} = \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{pmatrix} \tag{81}$$

Obviously $A\vec{v_3} = \sigma_3 \vec{u_3}$ without any calculation since $\sigma_3 = 0$.

Thus our right eigenmatrix V is:

$$V = \begin{pmatrix} \frac{1}{3\sqrt{2}} & \frac{-1}{\sqrt{2}} & \frac{2}{3} \\ \frac{-4}{3\sqrt{2}} & 0 & \frac{1}{3} \\ \frac{1}{3\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{2}{3} \end{pmatrix} \tag{82}$$

$$V^T = \begin{pmatrix} \frac{1}{3\sqrt{2}} & \frac{-4}{3\sqrt{2}} & \frac{1}{3\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{pmatrix} \tag{83}$$

We finally confirm that the decomposition works:

$$\begin{aligned} U\Sigma V^T &= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{3\sqrt{2}} & \frac{-4}{3\sqrt{2}} & \frac{1}{3\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{pmatrix} \\ &= \begin{pmatrix} \frac{3}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{3}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{3\sqrt{2}} & \frac{-4}{3\sqrt{2}} & \frac{1}{3\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{pmatrix} \\ &= \begin{pmatrix} 1 & -2 & 0 \\ 0 & -2 & 1 \end{pmatrix} \\ &= A \end{aligned} \tag{84}$$

4.1 Revisiting the choice of eigenvector issue

Recall that in this example we could have chosen perfectly legitimate orthonormal eigenvectors and formed the alternative left eigenmatrix:

$$\Omega = \begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix} \tag{85}$$

What happens if we use this matrix? It should be the case that $AV = \Omega\Sigma$ so let's check:

$$\begin{aligned} AV &= \begin{pmatrix} 1 & -2 & 0 \\ 0 & -2 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{3\sqrt{2}} & \frac{-1}{\sqrt{2}} & \frac{2}{3} \\ \frac{-4}{3\sqrt{2}} & 0 & \frac{1}{3} \\ \frac{1}{3\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{2}{3} \end{pmatrix} \\ &= \begin{pmatrix} \frac{3\sqrt{2}}{2} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{3\sqrt{2}}{2} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \end{aligned} \tag{86}$$

$$\begin{aligned} \Omega\Sigma &= \begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \frac{-3}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{-3}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \end{pmatrix} \\ &= \begin{pmatrix} \frac{-3\sqrt{2}}{2} & \frac{1}{\sqrt{2}} & 0 \\ \frac{-3\sqrt{2}}{2} & \frac{-1}{\sqrt{2}} & 0 \end{pmatrix} \end{aligned} \tag{87}$$

Notice that $\Omega\Sigma = -AV$ which just reflects the fact that the alternative eigenvectors are related thus $\vec{\mu_i} = -\vec{u_i}$ (see discussion around (71)). With this knowledge you should be able to predict what $\Omega\Sigma V^T$ is:

$$\begin{aligned} \Omega\Sigma V^T &= \begin{pmatrix} \frac{-3}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{-3}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{3\sqrt{2}} & \frac{-4}{3\sqrt{2}} & \frac{1}{3\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{pmatrix} \\ &= \begin{pmatrix} -1 & 2 & 0 \\ 0 & 2 & -1 \end{pmatrix} \\ &= -A \end{aligned} \tag{88}$$

What was the problem with Ω ? The problem is not with Ω but with V^T. Ω reflects $-\vec{v_1}$ and $-\vec{v_2}$. Recall that in the earlier discussion around (35) that when we chose alternative $\vec{u_2}$ we could still get a consistent end result by choosing $-\vec{v_2}$ and the same applies here. If, instead of choosing $\vec{v_2} = \begin{pmatrix} \frac{-1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$ (see (79)), we choose its -1 multiple $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{-1}{\sqrt{2}} \end{pmatrix}$ and we also modify our new right eigenmatrix by inserting $-\vec{v_1}$ to get a new right eigenmatrix:

$$\hat{V} = \begin{pmatrix} \frac{-1}{3\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{2}{3} \\ \frac{4}{3\sqrt{2}} & 0 & \frac{1}{3} \\ \frac{-1}{3\sqrt{2}} & \frac{-1}{\sqrt{2}} & \frac{2}{3} \end{pmatrix} \tag{89}$$

Hence:

$$\hat{V}^T = \begin{pmatrix} \frac{-1}{3\sqrt{2}} & \frac{4}{3\sqrt{2}} & \frac{-1}{3\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{pmatrix} \tag{90}$$

So:

$$\begin{aligned} \Omega\Sigma\hat{V}^T = & \begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{-1}{3\sqrt{2}} & \frac{4}{3\sqrt{2}} & \frac{-1}{3\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{pmatrix} \\ = & \begin{pmatrix} 1 & -2 & 0 \\ 0 & -2 & 1 \end{pmatrix} \\ = & A \end{aligned} \tag{91}$$

All this is doing is demonstrating is that we have uniqueness up to sign.

Recall that in 2 dimensions a rotation matrix $R(\theta)$ has the following form:

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{92}$$

Our original U (see (71)) was :

$$U = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \tag{93}$$

which has the form of a conventional counterclockwise rotation by $\frac{\pi}{4}$ radians. Thus $U \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \sqrt{2} \end{pmatrix}$.

Note that:

$$R(-\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \tag{94}$$

In fact:

$$R(\frac{-3\pi}{4}) = \Omega \tag{95}$$

So U was based on the standard counter clockwise rotation by $\frac{\pi}{4}$ radians while Ω reflects a clockwise (or negative) rotation of $\frac{3\pi}{4}$ radians. It looks like Mathematica chooses the standard counter clockwise rotational format when working out the SVD using the command SingularValueDecomposition[svdexample] where svdexample= $\begin{pmatrix} 1 & -2 & 0 \\ 0 & -2 & 1 \end{pmatrix}$:

$$\left\{ \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{3\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{2}{3} \\ -\frac{2\sqrt{2}}{3} & 0 & \frac{1}{3} \\ \frac{1}{3\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{2}{3} \end{pmatrix} \right\}$$

The output in Mathematica is in the format (U, Σ, V) not (U, Σ, V^T)

4.2 Geometrical interpretation of the effect of V^T

The right eigenmatrix V^T is guaranteed by general theory to be an orthogonal matrix and hence one that gives rise to a rotation. In 3-space there are 3 standard rotation matrices which are naturally based on the x, y, z axes $(R_x(\theta), R_y(\theta), R_z(\theta))$:

$$R_x(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \tag{96}$$

$$R_y(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \tag{97}$$

$$R_z(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{98}$$

Does V^T look like any of these 3 standard rotation matrices in (96)-(98)? Recall (see (83)) that:

$$V^T = \begin{pmatrix} \frac{1}{3\sqrt{2}} & \frac{-4}{3\sqrt{2}} & \frac{1}{3\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{pmatrix} \tag{99}$$

It does not seem that V^T is one of those 3 special forms but it is a general property of rotations in 3 dimensions that we can write V^T as a product of the 3 standard orthogonal matrices (and even more generally one can add a fixed vector to account for rotations around an arbitrary axis). We know what V^T does to $\begin{pmatrix}1\\0\\0\end{pmatrix}$ so can we find suitable rotation matrices to achieve the same result?

$$V^T\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}\frac{1}{3\sqrt{2}}\\\frac{-1}{\sqrt{2}}\\\frac{2}{3}\end{pmatrix} \tag{100}$$

It turns out that:

$$R_z\Big(-\arccos\big(\frac{1}{\sqrt{10}}\big)\Big)R_y\Big(\arcsin(\frac{-2}{3})\Big)\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}\frac{1}{3\sqrt{2}}\\\frac{-1}{\sqrt{2}}\\\frac{2}{3}\end{pmatrix} \tag{101}$$

Writing the matrices out in full (see (97) and (98)):

$$R_z\Big(-\arccos\big(\frac{1}{\sqrt{10}}\big)\Big) = \begin{pmatrix}\frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} & 0\\\frac{-3}{\sqrt{10}} & \frac{1}{\sqrt{10}} & 0\\0 & 0 & 1\end{pmatrix} \tag{102}$$

and

$$R_y\Big(\arcsin\big(\frac{-2}{3}\big)\Big) = \begin{pmatrix}\frac{\sqrt{5}}{3} & 0 & \frac{-2}{3}\\0 & 1 & 0\\\frac{2}{3} & 0 & \frac{\sqrt{5}}{3}\end{pmatrix} \tag{103}$$

In relation to (102) note that:

$$\cos\Big[\pm\arccos\Big(\frac{1}{\sqrt{10}}\Big)\Big] = \frac{1}{\sqrt{10}} \tag{104}$$

and

$$\begin{aligned}-\sin\Big[-\arccos\Big(\frac{1}{\sqrt{10}}\Big)\Big] &= \sin\Big[\arccos\Big(\frac{1}{\sqrt{10}}\Big)\Big]\\&= \frac{3}{\sqrt{10}}\end{aligned} \tag{105}$$

(104) follows from the fact that if $\arccos\big(\frac{1}{\sqrt{10}}\big) = A$ then $\cos A = \frac{1}{\sqrt{10}}$ so $\sin A = \frac{3}{\sqrt{10}}$. In relation to (103) note that if $\arcsin\big(\frac{2}{3}\big) = A$ then $\sin A = \frac{2}{3}$ and so $\cos A = \frac{\sqrt{5}}{3}$.

Let's now apply (104) and (105) to $\begin{pmatrix}1\\0\\0\end{pmatrix}$:

$$\begin{pmatrix}\frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} & 0\\ \frac{-3}{\sqrt{10}} & \frac{1}{\sqrt{10}} & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\frac{\sqrt{5}}{3} & 0 & \frac{-2}{3}\\ 0 & 1 & 0\\ \frac{2}{3} & 0 & \frac{\sqrt{5}}{3}\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}\frac{1}{3\sqrt{2}} & \frac{3}{\sqrt{10}} & \frac{-2}{3\sqrt{10}}\\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{10}} & \frac{6}{\sqrt{10}}\\ \frac{2}{3} & 0 & \frac{\sqrt{5}}{3}\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix}$$
$$= \begin{pmatrix}\frac{1}{3\sqrt{2}}\\ \frac{-1}{\sqrt{2}}\\ \frac{2}{3}\end{pmatrix} \tag{106}$$

It is to be noted that we didn't need to use $R_x(\theta)$ and the reason for that is that I chose the vector being acted on to be the unit basis vector in the x direction ie it sits on the axis of rotation of $R_x(\theta)$ and it won't be changed under $R_x(\theta)$ since all the "action" is orthogonal to it! Thus for any θ:

$$R_x(\theta)\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\ 0 & \cos\theta & -\sin\theta\\ 0 & \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix} \tag{107}$$

Similarly if I had chosen the unit basis vectors in the y and z directions to be the vector acted on, those matrices would not have figured in the calculation (note how each unit basis vector is embedded in each respective rotation matrix).

So how were the angles worked out? All you need to do is solve (and you can do it by hand in a minute):

$$R_z(\gamma)R_y(\beta)R_x(\alpha)\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}\frac{1}{3\sqrt{2}}\\ \frac{-1}{\sqrt{2}}\\ \frac{2}{3}\end{pmatrix} \tag{108}$$

We know that we can drop $R_x(\alpha)$ so we only have to use the remaining two matrices and we get:

$$R_z(\gamma)R_y(\beta)\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}\cos\gamma\cos\beta\\ \sin\gamma\cos\beta\\ -\sin\beta\end{pmatrix} = \begin{pmatrix}\frac{1}{3\sqrt{2}}\\ \frac{-1}{\sqrt{2}}\\ \frac{2}{3}\end{pmatrix} \tag{109}$$

and (101) follows from (109) after simple calculations.

5 Conclusions

These examples have probed some of the detailed issues which the SVD decomposition throws up. Obviously, when the matrices are huge rectangular things we cannot visualise the rotation and stretching effects as easily as we can with the 2 and 3 dimensional cases but the theory guarantees the existence of orthogonal and diagonal matrices which reflect the low dimensional structural properties. The 2 and 3 dimensional elliptical analysis extends to higher dimensions where the "ellipses" are "hyper-ellipses". To appreciate this just thgink of an $n \times n$ positive definite matrix A and vectors $\vec{x}$ and $\vec{v}$. The ellipse is represented by $(\vec{x} - \vec{v})^T A(\vec{x} - \vec{v}) = 1$ (if $\vec{v} = \vec{0}$ the hyper-ellipse is centred on the origin). Both MATLAB and Mathematica will choose eigenvectors to ensure that the decomposition works as intended and as we have seen there is uniqueness up to sign.

6 History

Created 11 July 2019
22 July 2019 - inserted missing λ in (12)