Solution to a 1985 Putnam integration problem

Peter Haggstrom
gotohaggstrom.com
mathsatbondibeach@gmail.com

January 17, 2019

1 Introduction

I came across the following Putnam problem which piqued my interest because of its trigonometrical nature. The problem was as follows. Let $I_m = \int_0^{2\pi} \cos(x)\,\cos(2x)\dots\cos(mx)\,dx$. For which integers m, $1 \leq m \leq 10$ is $I_m \neq 0$?

2 Line of attack

Since we are looking at the product of cosines one approach is to note that:

$$\int_0^{2\pi} \cos(x)\,\cos(2x)\dots\cos(mx)\,dx = \int_0^{2\pi} \Big(\frac{e^{ix}+e^{-ix}}{2}\Big)\Big(\frac{e^{2ix}+e^{-2ix}}{2}\Big)\dots\Big(\frac{e^{mix}+e^{-mix}}{2}\Big)\,dx \tag{1}$$

In terms of overall strategy we may end up taking the real part of the RHS of (1) and we have to tame the product of exponentials. But if we finally get down to components like the one in (2) (where k is integral) we are basically home and hosed:

$$\int_0^{2\pi} e^{\pm kix}\,dx \tag{2}$$

Why? The integral in (2) is :

$$\begin{aligned}\int_0^{2\pi} e^{\pm kix}\,dx &= \int_0^{2\pi} \cos(\pm kx)\,dx + i\underbrace{\int_0^{2\pi} \sin(\pm kx)\,dx}_{=0 \quad \forall k} \\ &= \int_0^{2\pi} \cos(kx)\,dx\end{aligned} \tag{3}$$

Now in (3), if $k = 0$, $\cos(kx) = 1$ and the integral is 2π, while if $k \neq 0$ the integral is simply $\frac{1}{k}\sin(2k\pi) = 0$ for all integral k. This simple observation is the driver of the solution to this problem.

We now need to tame the RHS of (2) and we do this as follows:

$$\int_0^{2\pi} \Big(\frac{e^{ix}+e^{-ix}}{2}\Big)\Big(\frac{e^{2ix}+e^{-2ix}}{2}\Big)\dots\Big(\frac{e^{mix}+e^{-mix}}{2}\Big)\,dx = \int_0^{2\pi} \Big(\frac{1+e^{2ix}}{2e^{ix}}\Big)\Big(\frac{1+e^{4ix}}{2e^{2ix}}\Big)\dots\Big(\frac{1+e^{2mix}}{2e^{mix}}\Big)\,dx$$
$$= \frac{1}{2^m}\int_0^{2\pi} \frac{(1+e^{2ix})(1+e^{4ix})\dots(1+e^{2mix})}{e^{\frac{m(m+1)ix}{2}}}\,dx \tag{4}$$

Note here that $e^{ix+2ix+\dots mix} = e^{\frac{m(m+1)ix}{2}}$.

We now have a product in the integrand which has a known expansion:

$$\prod_{k=1}^{m}(1+a_k) = 1 + \sum_{k=1}^{m} a_k + \sum_{k<l}^{m} a_k a_l + \sum_{k<l<p}^{m} a_k a_l a_p + \dots + a_1 a_2 \dots a_m \tag{5}$$

In this case $a_k = e^{2kix}$.

So where does this get us? Well (5) gives us the numerator in (4) and it is simply a sum of a lot of poducts of exponentials all of which are divided by the term $e^{\frac{m(m+1)ix}{2}}$ which will in turn leave us with sum of exponentials which we know how to integrate. We only need to find cosine terms with a zero angle and we will have solved the problem. Now for a bit more detail.

Using (5) we have that:

$$\prod_{p_1=1}^{m}(1+e^{2p_1 ix}) = 1+\sum_{p_1=1}^{m} e^{2p_1 ix}+\sum_{p_1<p_2}^{m} e^{2(p_1+p_2)ix}+\sum_{p_1<p_2<p_3}^{m} e^{2(p_1+p_2+p_3)ix}+\dots+e^{m(m+1)ix} \tag{6}$$

Note that in the last term in (6) we have $e^{2(1+2+\dots+m)ix} = e^{2\frac{m}{2}(m+1)ix} = e^{m(m+1)ix}$.

Thus the intergrand in the RHS of (4) becomes, on using (6):

$$e^{-\frac{m(m+1)ix}{2}} + \sum_{p_1=1}^{m} e^{\left(2p_1 - \frac{m(m+1)}{2}\right)ix} + \sum_{p_1<p_2}^{m} e^{\left(2(p_1+p_2) - \frac{m(m+1)}{2}\right)ix}$$
$$+ \sum_{p_1<p_2<p_3}^{m} e^{\left(2(p_1+p_2+p_3) - \frac{m(m+1)}{2}\right)ix} + \dots + e^{\left(m(m+1) - \frac{m(m+1)}{2}\right)ix} \tag{7}$$

When we integrate (7) from 0 to 2π the first term is zero (remember the discussion of (2)) and so if we want $I_m \neq 0$ we must have all the angular exponents in (7) equal to zero because we will then get the integral of 1 for each of them since $\cos 0 = 1$. So in general we want:

$$2(p_1 + p_2 + \cdots + p_k) - \frac{m(m+1)}{2} = 0 \quad \text{for } 1 \leq k \leq m \tag{8}$$

So the condition we are after is that $\frac{m(m+1)}{2}$ is even (since $2(p_1 + p_2 + \cdots + p_k)$ is even) and all we have to do is construct the following table:

m	$\frac{m(m+1)}{2}$	I_m
1	1	0
2	3	0
3	6	$\neq 0$
4	10	$\neq 0$
5	15	0
6	21	0
7	28	$\neq 0$
8	36	$\neq 0$
9	45	0
10	55	0

So the values of of m for which $I_m \neq 0$ are $m = 3, 4, 7, 8$.

Here is the verification from Mathematica:

In[6]:= $\int_0^{2\pi} \text{Cos}[x]\, dx$

Out[6]= 0

In[7]:= $\int_0^{2\pi} \text{Cos}[x]\ \text{Cos}[2\,x]\, dx$

Out[7]= 0

In[8]:= $\int_0^{2\pi} \text{Cos}[x]\ \text{Cos}[2\,x]\ \text{Cos}[3\,x]\, dx$

Out[8]= $\frac{\pi}{2}$

In[10]:= $\int_0^{2\pi} \text{Cos}[x]\ \text{Cos}[2\,x]\ \text{Cos}[3\,x]\ \text{Cos}[4\,x]\, dx$

Out[10]= $\frac{\pi}{4}$

In[12]:= $\int_0^{2\pi} \text{Cos}[x]\ \text{Cos}[2\,x]\ \text{Cos}[3\,x]\ \text{Cos}[4\,x]\ \text{Cos}[5\,x]\, dx$

Out[12]= 0

In[13]:= $\int_0^{2\pi} \text{Cos}[x]\ \text{Cos}[2\,x]\ \text{Cos}[3\,x]\ \text{Cos}[4\,x]\ \text{Cos}[5\,x]\ \text{Cos}[6\,x]\, dx$

Out[13]= 0

In[14]:= $\int_0^{2\pi} \text{Cos}[x]\ \text{Cos}[2\,x]\ \text{Cos}[3\,x]\ \text{Cos}[4\,x]\ \text{Cos}[5\,x]\ \text{Cos}[6\,x]\ \text{Cos}[7\,x]\, dx$

Out[14]= $\frac{\pi}{8}$

In[15]:= $\int_0^{2\pi} \text{Cos}[x]\ \text{Cos}[2\,x]\ \text{Cos}[3\,x]\ \text{Cos}[4\,x]\ \text{Cos}[5\,x]\ \text{Cos}[6\,x]\ \text{Cos}[7\,x]\ \text{Cos}[8\,x]\, dx$

Out[15]= $\frac{7\,\pi}{64}$

In[20]:= $\int_0^{2\pi} \text{Cos}[x]\ \text{Cos}[2\,x]\ \text{Cos}[3\,x]\ \text{Cos}[4\,x]\ \text{Cos}[5\,x]\ \text{Cos}[6\,x]\ \text{Cos}[7\,x]\ \text{Cos}[8\,x]\ \text{Cos}[9\,x]\, dx$

Out[20]= 0

In[21]:= $\int_0^{2\pi} \text{Cos}[x]\ \text{Cos}[2\,x]\ \text{Cos}[3\,x]\ \text{Cos}[4\,x]\ \text{Cos}[5\,x]\ \text{Cos}[6\,x]\ \text{Cos}[7\,x]\ \text{Cos}[8\,x]\ \text{Cos}[9\,x]\ \text{Cos}[10\,x]\, dx$

Out[21]= 0

It is significanrly harder to work out a formula for the non-zero values since this involves working out the coefficients of the relevant terms in (4). This involves some combinatorial/generating function insights and inspiration can be had from books by Herbert Wilf and Laszlo Lovasz.

3 History

Created 17 January 2019