Solutions to Feynman-Hibbs classical action problems

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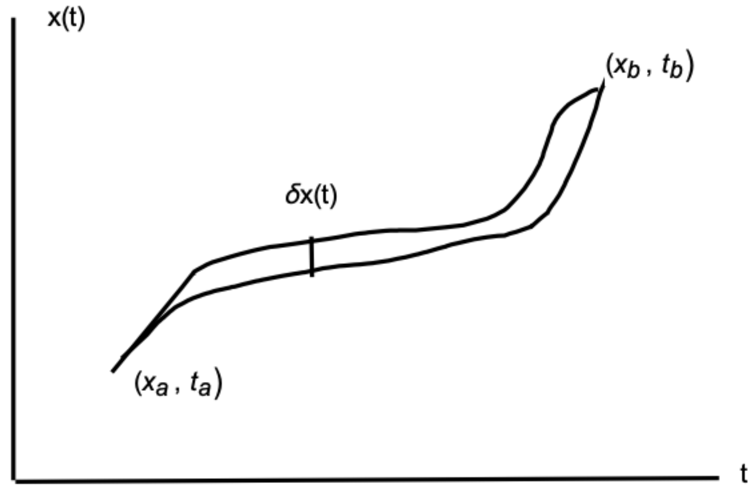
1 Introduction

Feyman developed his path integral approach to quantum mechanics in his PhD thesis and later he and Albert Hibbs produced a textbook on path integrals [1]. In this book, Feynman and Hibbs pose a series of fundamental problems which relate to the principle of least action. To set the scene for these problems they give a short derivation of the conditions for an extremum. I will simply reproduce that derivation here (see pages 26-27 of [1]). More detailed expositions of the calculus of variations concepts can be found in [2], [3].

The idea is to find the particular path $\overline{x}(t)$ out of all possible paths where a particle starts at point x_a at time t_a and ends at point x_b at time t_b. In classical theory there is only once such path but it quantum theory there are multiple trajectories each with a certain amplitude (represented by a kernel in the relevant integral representation) and the aim is to sum over all these trajectories between the start and end points.

In classical theory there is a quantity $\mathcal{S}$ which is associated with each path and the classical path $\overline{x}(t)$ is the one for which $\mathcal{S}$ is a minimum (more precisely an extremum). Thus if we perturb the path slightly the value of $\mathcal{S}$ is unchanged to first order.



The definition of $\mathcal{S}$ is as follows:

$$\mathcal{S} = \int_{t_a}^{t_b} \mathcal{L}(\dot{x}, x, t)\, dt \tag{1}$$

where $\mathcal{L}$ is the Lagrangian for the system. For a particle of mass m subject to potential energy $V(x,t)$ (ie a function of position and time) the Lagrangian is:

$$\mathcal{S} = \frac{m}{2}\dot{x}^2 - V(x,t) \tag{2}$$

We suppose that we vary the path $\overline{x}(t)$ a little by $\delta x(t)$ with condition that the end points be fixed ie:

$$\delta x(t_a) = \delta x(t_b) = 0 \tag{3}$$

For $\overline{x}(t)$ to be an extremum we must have:

$$\delta\mathcal{S} = \mathcal{S}[\overline{x} + \delta x] - \mathcal{S}[\overline{x}] = 0 \tag{4}$$

to the first order of $\delta x(t)$. Thus we have:

$$\begin{aligned}
\mathcal{S}[x+\delta x] &= \int_{t_a}^{t_b} \mathcal{L}(\dot{x}+\delta\dot{x}, x+\delta x, t)\,dt \\
&= \int_{t_a}^{t_b} \left[\mathcal{L}(\dot{x},x,t) + \delta x\frac{\partial \mathcal{L}}{\partial x} + \delta\dot{x}\frac{\partial \mathcal{L}}{\partial \dot{x}}\right] dt \\
&= \mathcal{S}[x] + \int_{t_a}^{t_b} \left[\delta x\frac{\partial \mathcal{L}}{\partial x} + \delta\dot{x}\frac{\partial \mathcal{L}}{\partial \dot{x}}\right] dt \\
&= \mathcal{S}[x] + \int_{t_a}^{t_b} \delta x\frac{\partial \mathcal{L}}{\partial x}\,dt + \left[\delta x\frac{\partial \mathcal{L}}{\partial \dot{x}}\right]_{t_a}^{t_b} - \int_{t_a}^{t_b} \delta x\,\frac{d}{dt}\Big(\frac{\partial \mathcal{L}}{\partial \dot{x}}\Big)\,dt \quad \text{using integration by parts} \\
&= \mathcal{S}[x] + \left[\delta x\frac{\partial \mathcal{L}}{\partial \dot{x}}\right]_{t_a}^{t_b} - \int_{t_a}^{t_b} \delta x\,\Big[\frac{d}{dt}\Big(\frac{\partial \mathcal{L}}{\partial \dot{x}}\Big) - \frac{\partial \mathcal{L}}{\partial x}\Big]\,dt \\
\therefore\ \delta\mathcal{S} &= \left[\delta x\frac{\partial \mathcal{L}}{\partial \dot{x}}\right]_{t_a}^{t_b} - \int_{t_a}^{t_b} \delta x\,\Big[\frac{d}{dt}\Big(\frac{\partial \mathcal{L}}{\partial \dot{x}}\Big) - \frac{\partial \mathcal{L}}{\partial x}\Big]\,dt
\end{aligned} \tag{5}$$

But from (3) we must have that $\left[\delta x\frac{\partial \mathcal{L}}{\partial \dot{x}}\right]_{t_a}^{t_b} = 0$ so that the extremum is the curve along which the following condition is satisfied:

$$\frac{d}{dt}\Big(\frac{\partial \mathcal{L}}{\partial \dot{x}}\Big) - \frac{\partial \mathcal{L}}{\partial x} = 0 \tag{6}$$

More detailed verifications of this condition can be found at pages 9-17 of [2]. Feynman and Hibbs denote the extreme value of the classical action integral $\mathcal{S} = \int \mathcal{L}\,dt$ by $\mathcal{S}_{cl}$.

2 Problem 2-1

For a free particle $\mathcal{L} = \frac{m}{2}\dot{x}^2$. Show that the action $\mathcal{S}_{cl}$ corresponding to the classical motion of a free particle is:

$$\mathcal{S}_{cl} = \frac{m}{2}\frac{(x_b - x_a)^2}{t_b - t_a} \tag{7}$$

2.1 Solution to Problem 2-1

Using condition (6) we see that:

$$\frac{d}{dt}\Big(\frac{\partial \mathcal{L}}{\partial \dot{x}}\Big) - \frac{\partial \mathcal{L}}{\partial x} = m\ddot{x} = 0 \tag{8}$$

So $\ddot{x} = 0$ which implies that $x(t) = xt + d$. With $x(t_a) = x_a$ and $x(t_b) = x_b$ we have that:

$$x(t) = \left(\frac{x_b - x_a}{t_b - t_a}\right) t + x_a t_b - x_b t_a \tag{9}$$

On this path the velocity $v(t)$ is constant and is simply $\frac{x_b - x_a}{t_b - t_a}$.

The classical action is given by:

$$\begin{aligned}
\mathcal{S}_{cl} &= \int_{t_a}^{t_b} \frac{m}{2}\dot{x}^2\, dt \\
&= \frac{m}{2}\int_{t_a}^{t_b} v(t)v(t)\, dt \\
&= \frac{m}{2}\left[\Big[v(t)x(t)\Big]_{t_a}^{t_b} - \int_{t_a}^{t_b} x(t)dv(t)\right] \\
&= \frac{m}{2}\left[\Big[v(t)x(t)\Big]_{t_a}^{t_b} - \int_{t_a}^{t_b} x(t)\ddot{x}(t)\, dt\right] \\
&= \frac{m}{2}\Big[v(t_b)x(t_b) - v(t_a)x(t_a)\Big] \quad \text{since } \ddot{x}(t) = 0 \\
&= \frac{m}{2}\Big[\Big(\frac{x_b - x_a}{t_b - t_a}\Big)x(t_b) - \Big(\frac{x_b - x_a}{t_b - t_a}\Big)x(t_a)\Big] \\
&= \frac{m}{2}\frac{(x_b - x_a)^2}{t_b - t_a}
\end{aligned} \tag{10}$$

3 Problem 2-2

For a harmonic oscillator $\mathcal{L} = \frac{m}{2}(\dot{x}^2 - \omega^2 x^2)$. With T equal to $t_b - t_a$ show that the classical action is:

$$\mathcal{S}_{cl} = \frac{m\omega}{2\sin\omega T}\big[(x_b^2 + x_a^2)\cos\omega T - 2x_b x_a\big] \tag{11}$$

3.1 Solution to Problem 2-2

Using condition (6) we see that:

$$\frac{d}{dt}(m\dot{x}) + m\omega^2 x = m\ddot{x} + m\omega^2 x = 0 \implies \ddot{x} + \omega^2 x = 0 \tag{12}$$

Hence a solution to (12) is given by:

$$x(t) = A\sin\omega t + B\cos\omega t \tag{13}$$

Note that $t^{'} = t - t_a$ is also a solution to (12). With $t = t_a$, $t^{'} = 0$ and $x(0) = x_a$ and with $t = t_b$, $t^{'} = t_b - t_a$ and $x(T) = x_b$. Therefore

$$x(0) = x_a = B \tag{14}$$

Hence:

$$\begin{aligned} x_b = X(T) =& A\sin\omega T + B\cos\omega T \\ =& A\sin\omega T + x_a\cos\omega T \\ \therefore A =& \frac{x_b - x_a\cos\omega T}{\sin\omega T} \end{aligned} \tag{15}$$

With $t \mapsto t - t_a$ and $T = t_b - t_a$ we have:

$$\begin{aligned} \mathcal{S}_{cl} =& \frac{m}{2}\int_0^T (\dot{x}^2 - \omega^2 x^2)\,dt \\ =& \frac{m}{2}\int_0^T v^2(t)\,dt - \frac{m}{2}\int_0^T \omega^2 x^2\,dt \\ =& \frac{m}{2}\left[\Big[v(t)x(t)\Big]_0^T - \int_0^T x(t)\ddot{x}(t)\,dt\right] - \frac{m}{2}\int_0^T \omega^2 x^2\,dt \quad \text{using (10) for the first integral} \\ =& \frac{m}{2}\left[\Big[v(t)x(t)\Big]_0^T - \int_0^T x(t)(-\omega^2 x)\,dt\right] - \frac{m}{2}\int_0^T \omega^2 x^2\,dt \quad \text{using (12) for the first integral} \\ =& \frac{m}{2}\big[v(T)x(T) - v(0)x(0)\big] \\ =& \frac{m}{2}\big[\dot{x}(T)x(T) - \dot{x}(0)x(0)\big] \end{aligned} \tag{16}$$

But from (13)-(15):

$$x(t) = \left(\frac{x_b - x_a\cos\omega T}{\sin\omega T}\right)\sin\omega t + x_a\cos\omega t \tag{17}$$

Hence:

$$\dot{x}(T) = \left(\frac{x_b - x_a\cos\omega T}{\sin\omega T}\right)\omega\cos\omega T - x_a\omega\sin\omega T \tag{18}$$

and

$$\dot{x}(0) = \left(\frac{x_b - x_a\cos\omega T}{\sin\omega T}\right)\omega \tag{19}$$

Putting it all together in (16) we have (noting that $x(T) = x_b$ and $x(0) = x_a$):

$$\begin{aligned}
\mathcal{S}_{cl} =& \frac{m}{2}\left[\dot{x}(T)x(T) - \dot{x}(0)x(0)\right] \\
=& \frac{m}{2}\left[\omega x_b\left(\frac{x_b - x_a\cos\omega T}{\sin\omega T}\right)\cos\omega T - x_a x_b\omega\sin\omega T - x_a\omega\left(\frac{x_b - x_a\cos\omega T}{\sin\omega T}\right)\right] \\
=& \frac{m\omega}{2\sin\omega T}\left[x_b(x_b - x_a\cos\omega T)\cos\omega T - x_a x_b\sin^2\omega T - x_a(x_b - x_a\cos\omega T)\right] \\
=& \frac{m\omega}{2\sin\omega T}\left[x_b^2\cos\omega T - x_a x_b\cos^2\omega T - x_a x_b\sin^2\omega T - x_a x_b + x_a^2\cos\omega T\right] \\
=& \frac{m\omega}{2\sin\omega T}\left[(x_b^2 + x_a^2)\cos\omega T - 2x_a x_b\right]
\end{aligned} \tag{20}$$

4 Problem 2-3

Find $\mathcal{S}_{cl}$ for a particle under a constant force f, that is, $\mathcal{L} = \frac{m}{2}\dot{x}^2 + fx$.

4.1 Solution to Problem 2-3

Using condition (6) we see that:

$$m\ddot{x} = f \tag{21}$$

Thus:

$$x(t) = \frac{f}{2m}t^2 + ct + d \tag{22}$$

is a solution to (21) and as in Problem 2-2 we note that $x(t - t_a)$ is also a solution to (21). We have from (22):

$$x(t_a - t_a) = x(0) = x_a = d \tag{23}$$

and

$$\begin{aligned}
x(t_b - t_a) =& x_b = \frac{f}{2m}(t_b - t_a)^2 + c(t_b - t_a) + x_a \\
\therefore c =& \frac{x_b - x_a}{t_b - t_a} - \frac{f}{2m}(t_b - t_a)
\end{aligned} \tag{24}$$

Hence since $x(t - t_a)$ is a solution to (21) we have with $\tau = t - t_a$:

$$x(\tau) = \frac{f}{2m}\tau^2 + \left[\frac{x_b - x_a}{t_b - t_a} - \frac{f}{2m}(t_b - t_a)\right]\tau + x_a \tag{25}$$

Therefore:

$$\dot{x}(\tau) = \frac{f\tau}{m} + \frac{x_b - x_a}{t_b - t_a} - \frac{f}{2m}(t_b - t_a) \tag{26}$$

Hence we have that:

$$\begin{aligned}
\mathcal{L}(\tau) =& \frac{m}{2}\dot{x}^2(\tau) + fx(\tau) \\
=& \frac{m}{2}\Big[\frac{f\tau}{m} + \frac{x_b - x_a}{t_b - t_a} - \frac{f}{2m}(t_b - t_a)\Big]^2 + \frac{f^2\tau^2}{2m} + f\tau\Big[\frac{x_b - x_a}{t_b - t_a} - \frac{f}{2m}(t_b - t_a)\Big] + x_a f \\
=& \frac{m}{2}\Big[\frac{f^2\tau^2}{m^2} + \Big(\frac{x_b - x_a}{t_b - t_a} - \frac{f}{2m}(t_b - t_a)\Big)^2 + \frac{2f\tau}{m}\Big(\frac{x_b - x_a}{t_b - t_a} - \frac{f}{2m}(t_b - t_a)\Big)\Big] + \\
& \frac{f^2\tau^2}{2m} + f\tau\Big[\frac{x_b - x_a}{t_b - t_a} - \frac{f}{2m}(t_b - t_a)\Big] + x_a f \\
=& \frac{f^2\tau^2}{m} + \frac{m}{2}\Big[\frac{x_b - x_a}{t_b - t_a} - \frac{f}{2m}(t_b - t_a)\Big]^2 + 2f\tau\Big[\frac{x_b - x_a}{t_b - t_a} - \frac{f}{2m}(t_b - t_a)\Big] + fx_a
\end{aligned} \tag{27}$$

The classical action is thus given by:

$$\begin{aligned}
\mathcal{S}_{cl} =& \int_0^{t_b - t_a}\left\{\frac{f^2\tau^2}{m} + \frac{m}{2}\Big[\frac{x_b - x_a}{t_b - t_a} - \frac{f}{2m}(t_b - t_a)\Big]^2 + 2f\tau\Big[\frac{x_b - x_a}{t_b - t_a} - \frac{f}{2m}(t_b - t_a)\Big] + fx_a\right\}d\tau \\
=& \left[\frac{f^2\tau^3}{3m} + \frac{m}{2}\Big[\frac{x_b - x_a}{t_b - t_a} - \frac{f}{2m}(t_b - t_a)\Big]^2\tau + f\Big[\frac{x_b - x_a}{t_b - t_a} - \frac{f}{2m}(t_b - t_a)\Big]\tau^2 + fx_a\tau\right]_0^{t_b - t_a} \\
=& \frac{f^2(t_b - t_a)^3}{3m} + \frac{m}{2}\Big[\frac{x_b - x_a}{t_b - t_a} - \frac{f}{2m}(t_b - t_a)\Big]^2(t_b - t_a) + f\Big[\frac{x_b - x_a}{t_b - t_a} - \frac{f}{2m}(t_b - t_a)\Big](t_b - t_a)^2 + fx_a(t_b - t_a) \\
=& \frac{f^2(t_b - t_a)^3}{3m} + \frac{m}{2}\frac{(x_b - x_a)^2}{(t_b - t_a)} - \frac{f}{2}(x_b - x_a)(t_b - t_a) + \frac{f^2}{8m}(t_b - t_a)^3 + f(x_b - x_a)(t_b - t_a) \\
& - \frac{f^2}{2m}(t_b - t_a)^3 + fx_a(t_b - t_a) \\
=& \frac{-f^2}{24m}(t_b - t_a)^3 + \frac{m}{2}\frac{(x_b - x_a)^2}{(t_b - t_a)} + \frac{f}{2}(x_b - x_a)(t_b - t_a) + fx_a(t_b - t_a) \\
=& \frac{-f^2}{24m}(t_b - t_a)^3 + \frac{m}{2}\frac{(x_b - x_a)^2}{(t_b - t_a)} + f(t_b - t_a)\Big[x_a + \frac{x_b - x_a}{2}\Big] \\
=& \frac{-f^2}{24m}(t_b - t_a)^3 + \frac{m}{2}\frac{(x_b - x_a)^2}{(t_b - t_a)} + \frac{f(x_a + x_b)(t_b - t_a)}{2} \\
=& \frac{-f^2}{24m}T^3 + \frac{m}{2}\frac{(x_b - x_a)^2}{T} + \frac{f(x_a + x_b)T}{2} \qquad \text{with } T = t_b - t_a
\end{aligned} \tag{28}$$

5 Problem 2-4

Classically, the momentum is defined as:

$$p = \frac{\partial \mathcal{L}}{\partial \dot{x}} \tag{29}$$

Show that the momentum at a final point is:

$$\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right)_{x=x_b} = +\frac{\partial \mathcal{S}_{cl}}{\partial x_b} \tag{30}$$

while the momentum at an initial point is:

$$\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right)_{x=x_a} = -\frac{\partial \mathcal{S}_{cl}}{\partial x_a} \tag{31}$$

Hint: Consider the effect on (5) of a change in the end points.

5.1 Solution to Problem 2-4

In equation (5), t_a is fixed so $\delta x(t_a) = 0$ but $\delta x(t_b) = \delta x_b \neq 0$ since we are varying x_b. This means that while the value of the quantity in (32) vanishes at the initial point there is a contribution from the variation in the final point:

$$\delta \mathcal{S}_{cl} = \left[\delta x \frac{\partial \mathcal{L}}{\partial \dot{x}}\right]_{t_a}^{t_b} = \delta x_b \times \frac{\partial \mathcal{L}}{\partial \dot{x}}\Big|_{t_b} - 0 \tag{32}$$

Hence we can write in the limit:

$$+\frac{\partial \mathcal{S}_{cl}}{\partial x_b} = \left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right)_{x=x_b} \tag{33}$$

When we keep the final point fixed with $\delta x(t_b) = 0$ and vary the initial point we get:

$$\delta \mathcal{S}_{cl} = \left[\delta x \frac{\partial \mathcal{L}}{\partial \dot{x}}\right]_{t_a}^{t_b} = -\delta x_a \times \frac{\partial \mathcal{L}}{\partial \dot{x}}\Big|_{t_a} \tag{34}$$

Hence we can write in the limit:

$$-\frac{\partial \mathcal{S}_{cl}}{\partial x_b} = \left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right)_{x=x_a} \tag{35}$$

We can check the logic of these results by applying them to Problems 2-1 and 2-2. For example, in Problem 2-1 we have that:

$$\frac{\partial \mathcal{S}_{cl}}{\partial x_b} = m\frac{(x_b - x_a)}{t_b - t_a} \tag{36}$$

which is clearly a momentum since on the classical path the velocity is a constant $\frac{x_b - x_a}{t_b - t_a}$. Furthermore from (9) we have that:

$$x(t) = \left(\frac{x_b - x_a}{t_b - t_a}\right)t + x_a t_b - x_b t_a \tag{37}$$

so that

$$\dot{x}(t) = \frac{x_b - x_a}{t_b - t_a} \tag{38}$$

Thus:

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x} = m\frac{(x_b - x_a)}{t_b - t_a} \tag{39}$$

noting that there is no explicit dependence on x for the evaluation of $\frac{\partial \mathcal{L}}{\partial \dot{x}}$ at $x = x_b$.

With Problem 2-2 we have that:

$$\frac{\partial \mathcal{S}_{cl}}{\partial x_b} = \frac{m\omega}{2\sin\omega T}\big(2x_b\cos\omega T - 2x_a\big) = \frac{m\omega}{\sin\omega T}\big(x_b\cos\omega T - x_a\big) \tag{40}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x} \tag{41}$$

From (17) we have that:

$$\begin{aligned}\dot{x}(t) = &\Big(\frac{x_b - x_a\cos\omega T}{\sin\omega T}\Big)\omega\cos\omega T - x_a\omega\sin\omega T \\ = &\,\omega\Big(\frac{x_b\cos\omega T - x_a}{\sin\omega T}\Big)\end{aligned} \tag{42}$$

Hence:

$$m\dot{x}(t) = m\omega\Big(\frac{x_b\cos\omega T - x_a}{\sin\omega T}\Big) = \frac{\partial \mathcal{S}_{cl}}{\partial x_b} \tag{43}$$

Again there is no explicit dependence on x for evaluation of $\frac{\partial \mathcal{L}}{\partial \dot{x}}$ at $x = x_b$.

6 Problem 2-5

Classically the energy is defined as:

$$E = \dot{x}p - \mathcal{L} \tag{44}$$

Show that the energy at a final point is:

$$\dot{x}_b \Big(\frac{\partial \mathcal{L}}{\partial \dot{x}}\Big)_{x=x_b} - \mathcal{L}(x_b) = -\frac{\partial \mathcal{S}_{cl}}{\partial t_b} \tag{45}$$

while the energy at an initial point is:

$$+\frac{\partial \mathcal{S}_{cl}}{\partial t_a} \tag{46}$$

Hint: A change in the time of an end point requires a change in path, since all paths must be classical paths.

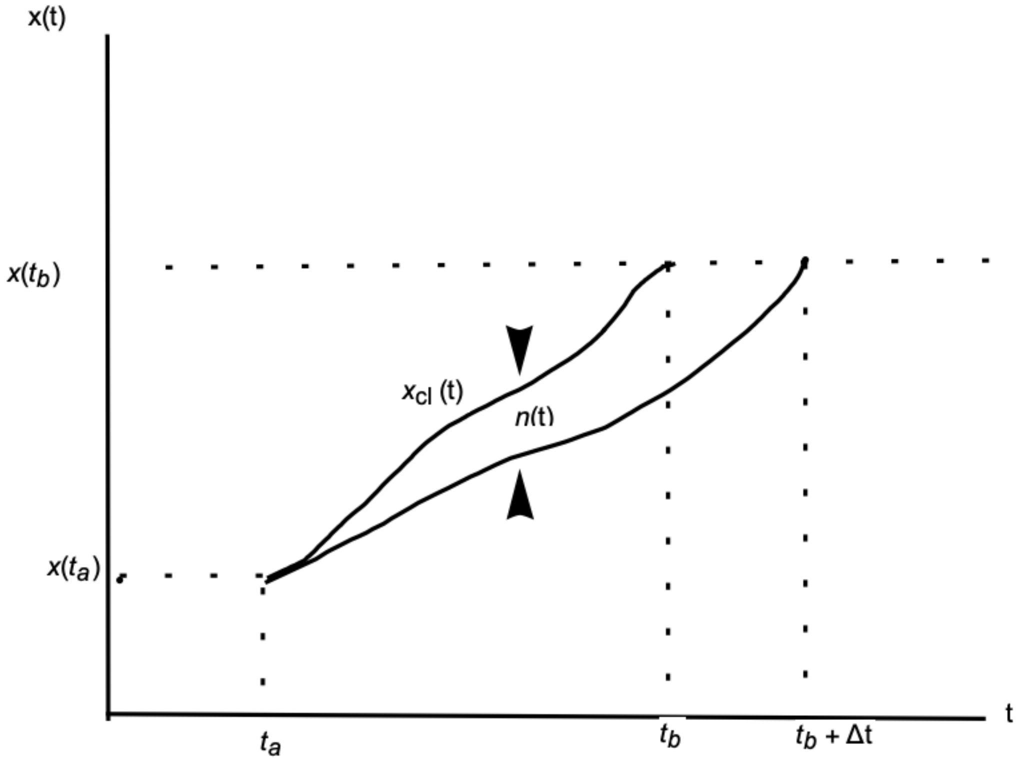
6.1 Solution to Problem 2-5

A naive approach to this problem might run this way. Since $\mathcal{S}_{cl} = \int_{t_a}^{t_b} \mathcal{L}(\dot{x}, x, t)\, dt$ one might argue that:

$$\begin{aligned} \frac{\partial \mathcal{S}_c}{\partial t} =& \frac{\partial}{\partial t} \int_{t_a}^{t_b} \mathcal{L}(\dot{x}, x, t)\, dt \\ =& \int_{t_a}^{t_b} \frac{\partial \mathcal{L}(\dot{x}, x, t)}{\partial t}\, dt \\ =& \mathcal{L}(\dot{x}, x, t_b) - \mathcal{L}(\dot{x}, x, t_a) \end{aligned} \tag{47}$$

which is not $-E$ ie negative energy.

A diagram helps:



There is a change in the classical path caused by the change in time of the path while holding the initial endpoints x_a and x_b fixed. Note that $\eta(t_a) = 0$ but $\eta(t_b) \neq 0$. The total change in the action involves two components: first, the bit of the classical path between t_a and t_b and then, secondly, the bit from t_b to $t_b + \Delta t$. The new path can be represented by:

$$x(t) = x_{cl}(t) + \eta(t) \tag{48}$$

subject to $\eta(t_a) = 0$.

We now go through the process of working out $\delta\mathcal{S}_{cl}$ where $\eta(t)$ is the deviation function. In terms of the action integral we have:

$$\mathcal{S}_{cl} = \int_{t_a}^{t_b+\Delta t} \mathcal{L}\, dt = \int_{t_a}^{t_b} \mathcal{L}\, dt + \int_{t_b}^{t_b+\Delta t} \mathcal{L}\, dt = \mathcal{S}_1 + \mathcal{S}_2 \tag{49}$$

So that:

$$\delta\mathcal{S}_{cl} = \delta\mathcal{S}_1 + \delta\mathcal{S}_2 \tag{50}$$

In working out $\delta\mathcal{S}$ will deal with the two integrals on the RHS of (49) separately:

$$\begin{aligned}
\delta\mathcal{S}_1 &= \int_{t_a}^{t_b} \mathcal{L}\big(\dot{x}_{cl}(t)+\dot{\eta}(t), x_{cl}(t)+\eta(t), t\big)\,dt - \int_{t_a}^{t_b} \mathcal{L}\big(\dot{x}_{cl}(t), x_{cl}(t), t\big)\,dt \\
&= \int_{t_a}^{t_b} \Big(\mathcal{L}\big(\dot{x}_{cl}(t), x_{cl}(t), t\big) + \eta(t)\frac{\partial \mathcal{L}}{\partial x} + \dot{\eta}(t)\frac{\partial \mathcal{L}}{\partial \dot{x}}\Big)\,dt - \int_{t_a}^{t_b} \mathcal{L}\big(\dot{x}_{cl}(t), x_{cl}(t), t\big)\,dt \\
&= \int_{t_a}^{t_b} \Big(\eta(t)\frac{\partial \mathcal{L}}{\partial x} + \dot{\eta}(t)\frac{\partial \mathcal{L}}{\partial \dot{x}}\Big)_{cl}\,dt \\
&= \int_{t_a}^{t_b} \eta(t)\frac{\partial \mathcal{L}}{\partial x}\,dt + \Big[\eta(t)\frac{\partial \mathcal{L}}{\partial \dot{x}}\Big]_{t_a}^{t_b} - \int_{t_a}^{t_b} \eta(t)\,\frac{d}{dt}\Big(\frac{\partial \mathcal{L}}{\partial \dot{x}}\Big)\,dt \\
&= \underbrace{\int_{t_a}^{t_b} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}} - \frac{d}{dt}\Big(\frac{\partial \mathcal{L}}{\partial \dot{x}}\Big)\right)\eta(t)\,dt}_{=0 \text{ since this is a classical path}} + \Big[\eta(t)\frac{\partial \mathcal{L}}{\partial \dot{x}}\Big]_{t_a}^{t_b} \\
&= \Big[\eta(t)\frac{\partial \mathcal{L}}{\partial \dot{x}}\Big]_{t_b} \quad \text{since } \eta(t_a)=0
\end{aligned} \tag{51}$$

Now for $\delta\mathcal{S}_2$, since the end points are fixed, the Lagrangian can be taken as fixed at time t_b and the variation is due to the extra travel between t_b and $t_b + \Delta t$. If we had any continuous function $f(x)$ we could estimate the integral $\int_a^{a+\Delta t} f(t)\,dt$ by $\Delta t \times f(a)$ where Δt is very small. In essence this is a continuity argument since for t very close to a, $f(t)$ is very close to $f(a)$. Thus:

$$\delta\mathcal{S}_2 = \int_{t_b}^{t_b+\Delta t} \mathcal{L}(\dot{x}_{cl}(t_b), x_{cl}(t_b), t_b)\,dt = \Delta t\,\mathcal{L}(\dot{x}_{cl}(t_b), x_{cl}(t_b), t_b) = \Delta t\,\mathcal{L}(t_b) \tag{52}$$

Hence:

$$\delta\mathcal{S}_{cl} = \Big[\eta(t)\frac{\partial \mathcal{L}}{\partial \dot{x}}\Big]_{t_b} + \Delta t\,\mathcal{L}(t_b) \tag{53}$$

Now we can get an expression for $\eta(t_b)$ as follows with the help of the diagram above, noting that the variation $\eta(t)$ is negative and the gradient at $x_{cl}(t_a)$ is positive. Thus:

$$x_{cl}(t_b + \Delta t) - x_{cl}(t_b) = -\eta(t_b) \tag{54}$$

Therefore:

$$\dot{x}_{cl}(t_b) = -\frac{\eta(t_b)}{\Delta t} \tag{55}$$

or

$$\eta(t_b) = -\dot{x}_{cl}(t_b)\,\Delta t \tag{56}$$

Plugging (56) into (53) we have:

$$\begin{aligned}\delta\mathcal{S}_{cl} =& \Big[-\dot{x}_{cl}(t_b)\,\Delta t \frac{\partial \mathcal{L}}{\partial \dot{x}}\Big]_{t_b} + \Delta t\,\mathcal{L}(t_b) \\ =& \Big[\mathcal{L} - \dot{x}_{cl}\frac{\partial \mathcal{L}}{\partial \dot{x}}\Big]_{t_b} \Delta t\end{aligned} \tag{57}$$

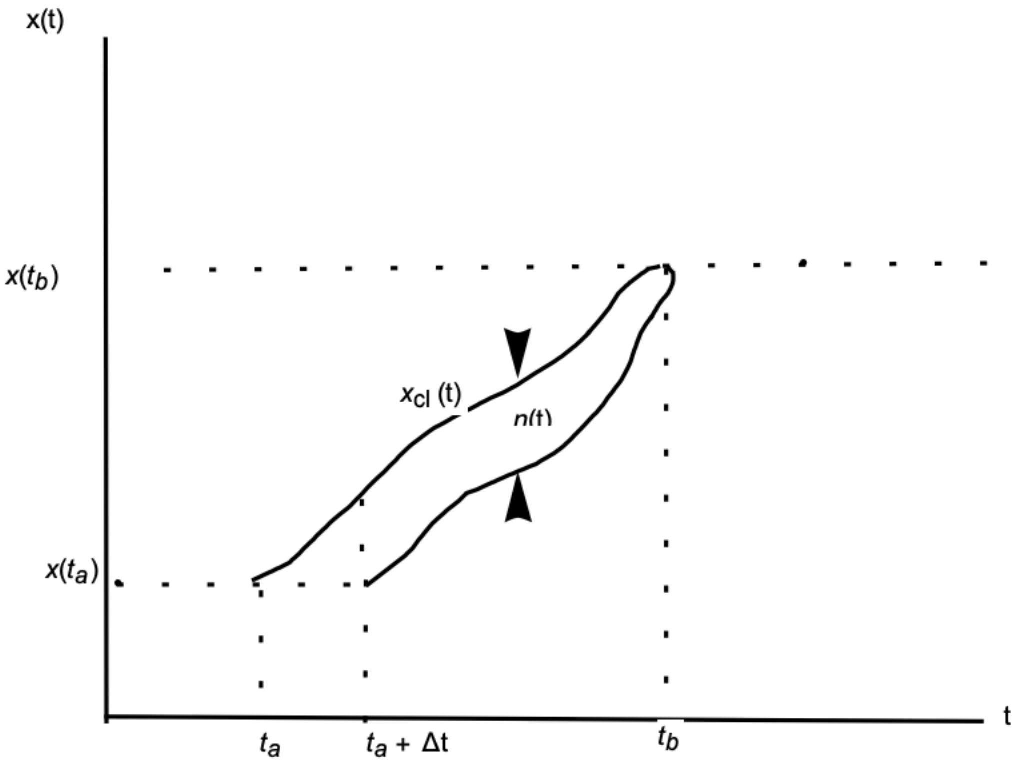
Finally we get:

$$\frac{\partial \mathcal{S}_{cl}}{\partial t_b} = \Big[\mathcal{L} - \dot{x}_{cl}\frac{\partial \mathcal{L}}{\partial \dot{x}}\Big]_{t_b} \tag{58}$$

whch is the same as (45) on multiplying both sides of (58) by -1. Equation (58) can be expressed in the Hamiltonian formulation as:

$$\frac{\partial \mathcal{S}_{cl}}{\partial t_b} = -\mathcal{H}(t_b) \tag{59}$$

To calculate the second part of the question we consider this diagram:



This time (49) becomes:

$$\mathcal{S}_{cl} = \int_{t_a+\Delta t}^{t_b} \mathcal{L}\,dt = \int_{t_a}^{t_b} \mathcal{L}\,dt - \int_{t_a}^{t_a+\Delta t} \mathcal{L}\,dt = \mathcal{S}_1 - \mathcal{S}_2 \tag{60}$$

We approximate $\delta\mathcal{S}_2$ as before (ie see (52)):

$$\delta\mathcal{S}_2 = \Delta t\,\mathcal{L}(t_a) \tag{61}$$

For completeness the derivation of $\delta\mathcal{S}_1$ is set out below although one could go straight to it because it follows the same lines as before:

$$\begin{aligned}
\delta\mathcal{S}_1 &= \int_{t_a}^{t_b} \mathcal{L}\big(\dot{x}_{cl}(t) + \dot{\eta}(t), x_{cl}(t) + \eta(t), t\big)\,dt - \int_{t_a}^{t_b} \mathcal{L}\big(\dot{x}_{cl}(t), x_{cl}(t), t\big)\,dt \\
&= \int_{t_a}^{t_b} \Big(\mathcal{L}\big(\dot{x}_{cl}(t), x_{cl}(t), t\big) + \eta(t)\frac{\partial\mathcal{L}}{\partial x} + \dot{\eta}(t)\frac{\partial\mathcal{L}}{\partial\dot{x}}\Big)\,dt - \int_{t_a}^{t_b} \mathcal{L}\big(\dot{x}_{cl}(t), x_{cl}(t), t\big)\,dt \\
&= \int_{t_a}^{t_b} \Big(\eta(t)\frac{\partial\mathcal{L}}{\partial x} + \dot{\eta}(t)\frac{\partial\mathcal{L}}{\partial\dot{x}}\Big)_{cl}\,dt \\
&= \int_{t_a}^{t_b} \eta(t)\frac{\partial\mathcal{L}}{\partial x}\,dt + \Big[\eta(t)\frac{\partial\mathcal{L}}{\partial\dot{x}}\Big]_{t_a}^{t_b} - \int_{t_a}^{t_b} \eta(t)\,\frac{d}{dt}\Big(\frac{\partial\mathcal{L}}{\partial\dot{x}}\Big)\,dt \\
&= \underbrace{\int_{t_a}^{t_b} \left(\frac{\partial\mathcal{L}}{\partial\dot{x}} - \frac{d}{dt}\Big(\frac{\partial\mathcal{L}}{\partial\dot{x}}\Big)\right)\eta(t)\,dt}_{=0 \text{ since this is a classical path}} + \Big[\eta(t)\frac{\partial\mathcal{L}}{\partial\dot{x}}\Big]_{t_a}^{t_b} \\
&= -\,\eta(t_a)\frac{\partial\mathcal{L}}{\partial\dot{x}}\Big]_{t_a} \quad \text{since } \eta(t_b) = 0
\end{aligned} \tag{62}$$

Thus:

$$\delta\mathcal{S}_{cl} = \Big[-\,\eta(t_a)\frac{\partial\mathcal{L}}{\partial\dot{x}}\Big]_{t_a} - \Delta t\,\mathcal{L}(t_a) \tag{63}$$

We have from the diagram (remembering that the variation $\eta(t)$ is negative and we actually have a positive derivative at t_a):

$$x_{cl}(t_a + \Delta t) - x_{cl}(t_a) = -\eta(t_a) \tag{64}$$

Therefore:

$$\dot{x}_{cl}(t_a) = -\frac{\eta(t_a)}{\Delta t} \tag{65}$$

or

$$-\,\eta(t_a) = \Delta t\,\dot{x}_{cl}(t_a) \tag{66}$$

Hence:

$$\begin{aligned}\delta\mathcal{S}_{cl} =& \Big[\Delta t\, \dot{x}_{t_a} \frac{\partial \mathcal{L}}{\partial \dot{x}}\Big]_{t_a} - \Delta t\, \mathcal{L}(t_a) \\ =& \Big[\dot{x}_{t_a} \frac{\partial \mathcal{L}}{\partial \dot{x}}\Big|_{t_a} - \mathcal{L}(t_a)\Big]\, \Delta t\end{aligned} \tag{67}$$

Finally we have:

$$\frac{\partial \mathcal{S}_{cl}}{\partial t_a} = \dot{x}_{t_a} \frac{\partial \mathcal{L}}{\partial \dot{x}}\Big|_{t_a} - \mathcal{L}(t_a) = \mathcal{H}(t_a) \tag{68}$$

7 References

[1] Richard P. Feynman and Albert R. Hibbs , *“Quantum Mechanics and Path Integrals”*, Emended by Daniel F. Styer, Dover, 2010

[2] I.M. Gelfand and S.V. Fomin, *“Calculus of Variations”,* translated and Edited by Richard A. Silverman, Dover, 2000.

[3] Cornelius Lanczos, *”The Variational Principles of Mechanics”, Fourth Edition* , Dover, 1970

8 History

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