Solving a quartic equation by the method of radicals

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1 Introduction

This paper deals with the solution of quartic equations by the method of radicals and builds upon an earlier paper and video which set out the process for solving a cubic by the method of radicals: `http://www.gotohaggstrom.com/All%20you%20wanted%20to%20know%20about%20solving%20cubics%20but%20were%20afraid%20to%20ask.pdf`

Accordingly you should read the cubic paper first because to get the general solution to a quartic you will need to solve a cubic along the way. The method of solution of the quartic by radicals (as explained in the cubic paper) is an intricate process originally devised by the 16^{th} century Italian mathematician Ferrari. One has to have admiration for Ferrari because the method is extremely prone to computational error as you will see. While the solution by radicals is really only of historical interest to students of number theory it nevertheless contains some interesting insights into how to solve a complicated problem using known sub-cases of a simpler problem.

So let's begin with the solution.

2 The recipe

2.1 Normalisation

$$a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 = 0$$
$$x^4 + ax^3 + bx^2 + cx + d = 0 \quad a = \frac{a_1}{a_0}, b = \frac{a_2}{a_0}, c = \frac{a_3}{a_0}, d = \frac{a_4}{a_0}$$

2.2 Linear transformation

$x = y + f$
$(y+f)^4 + a(y+f)^3 + b(y+f)^2 + c(y+f) + d = 0$ f unknown at this stage
$y^4 + (4f+a)y^3 + (6f^2+3af+b)y^2 + (4f^3+3af^2+2bf+c)y + (f^4+af^3+bf^2+cf+d) = 0.$ Now get rid of the cubed term.
Hence $4f + a = 0 \implies f = \frac{-a}{4}$
Thus, $x = y + f = x - \frac{a}{4} = x - \frac{a_1}{4a_0}$

2.3 Reduced quartic equation

We now have the reduced equation:
$y^4 + py^2 + qy + r = 0$ where p and q are polynomials in a,b,c: (2.3.1)
$p = 6f^2 + 3af + b$
$q = 4f^3 + 3af^2 + 2fb + c$
$r = f^4 + af^3 + bf^2 + cf + d$

To solve $y^4 + py^2 + qy + r = 0$ we use a form of completing the square invented by Ferrari.

2.4 Completing the square in (2.3.1)

Completing the square in (2.3.1) we get:
$y^4 + py^2 + qy + r = (y^2 + \frac{p}{2})^2 + qy + (r - \frac{p^2}{4}) = 0$ (2.4.1)
We now introduce an arbitrary parameter α as follows:
$(y^2 + \frac{p}{2} + \alpha)^2 = (y^2 + \frac{p}{2})^2 + 2(y^2 + \frac{p}{2})\alpha + \alpha^2$
Hence, using (2.4.1) the following is true:
$(y^2 + \frac{p}{2} + \alpha)^2 - [2\alpha(y^2 + \frac{p}{2}) + \alpha^2 - qy + \frac{p^2}{4} - r] = 0$ (2.4.2)
The idea is to find α such that the following is a second order polynomial in y which in turn is a perfect square:
$2\alpha y^2 - qy + (\alpha p + \alpha^2 + \frac{p^2}{4} - r)$ (2.4.3)
The logic behind this is that if (2.4.3) is a perfect square then (2.4.2) becomes the difference of two squares $s^2 - t^2 = (s-t)(s+t) = 0$ which is the product of two polynomials of the second degree in y.

2.5 Necessary and sufficient condition for (2.4.3) to be a perfect square

We know that since (2.4.3) is a quadratic in y a necessary and sufficient condition for it to be a perfect square is that the discriminant be zero:
$\Delta = q^2 - 8\alpha(\alpha p + \alpha^2 + \frac{p^2}{4} - r) = -8\alpha^3 - 8p\alpha^2 - 8(\frac{p^2}{4} - r)\alpha + q^2 = 0$
ie:
$\alpha^3 + p\alpha^2 + (\frac{p^2}{4} - r)\alpha - \frac{q^2}{8} = 0 \quad (2.5.1)$
But (2.5.1) is a cubic which can be solved by the method of radicals and so one can take one such solution for α and plug it into (2.4.2) which will yield something of the form:
$(y^2 + \frac{p}{2} + \alpha)^2 - t(y)^2 = 0 \quad (2.5.2)$
where $t(y)$ is a perfect square because of the choice of α. Clearly (2.5.2) can be solved for y

2.6 Working our way back to the original problem

We now have to work our way back to the original problem systematically. This is extremely tedious because we have potentially complicated expressions for α . The best way to follow the process is via an example

2.7 Applying the recipe to an example

To see how the formula works in detail it is necessary to do an example. To this end I'll concoct an example whose analytical solution is easy if you use Mathematica or Matlab!:

$x^4 - 6x^3 + 4x^2 + 4x - 12 = 0 \quad (2.7.1)$

Mathematica gives the solutions as:

$$x_{1,2,3,4} = \begin{cases} 1 - i \\ 1 + i \\ 2 - \sqrt{10} \\ 2 + \sqrt{10} \end{cases} \qquad (2.7.2)$$

As before with the cubic we make a linear transformation as follows:

$x = y + d$ where d is to be determined (2.7.3)

We substitute $x = y + d$ into (2.7.1) and after expanding and collecting terms we get:

$y^4 + (4d - 6)y^3 + (6d^2 - 18d + 4)y^2 + (4d^3 - 18d^2 + 8d + 4)y + (d^4 - 6d^3 + 4d^2 + 4d - 12) = 0$ ...(2.7.4)

To get rid of the cubic term in (2.7.4) we need:

$4d - 6 = 0 \implies d = \frac{3}{2} \quad (2.7.5)$

Equation (2.7.4) becomes a 'reduced' quartic of the following form :

$y^4 + py^2 + qy + r = 0$

where (on plugging in d):

$p = 6d^2 - 18d + 4 = -\frac{19}{2} \quad (2.7.6)$

$q = 4d^3 - 18d^2 + 8d + 4 = -11 \quad (2.7.7)$

$r = d^4 - 6d^3 + 4d^2 + 4d - 12 = -\frac{195}{16} \quad (2.7.8)$

Hence we have:

$y^4 - \frac{19}{2}y^2 - 11y - \frac{195}{16} = 0 \quad (2.7.9)$

We need to modify (2.7.9) so that it is the difference of two squares so that we can solve for y and to do this we introduce a variable α to be determined:

$$y^4 - \frac{19}{2}y^2 - 11y - \frac{195}{16} = (y^2 - \frac{19}{4})^2 - \frac{361}{16} - 11y - \frac{195}{16} = (y^2 - \frac{19}{4})^2 - 11y - \frac{139}{4} = 0 \quad (2.7.10)$$

Hence:

$$(y^2 - \frac{19}{4} + \alpha)^2 = (y^2 - \frac{19}{4})^2 + 2\alpha(y^2 - \frac{19}{4}) + \alpha^2$$

$$= \underbrace{(y^2 - \frac{19}{4})^2 - 11y - \frac{139}{4}}_{=0 \text{ from } (2.7.10)} + 2\alpha(y^2 - \frac{19}{4}) + \alpha^2 + 11y + \frac{139}{4}$$

$$= 2\alpha(y^2 - \frac{19}{4}) + \alpha^2 + 11y + \frac{139}{4}$$

Therefore:

$$(y^2 - \frac{19}{4} + \alpha)^2 - \underbrace{[2\alpha(y^2 - \frac{19}{4}) + \alpha^2 + 11y + \frac{139}{4}]}_{\text{we want this to be a perfect square}} = 0 \quad (2.7.11)$$

Then we will have something of the form:
$s(y)^2 - t(y)^2 = (s(y) - t(y))(s(y) + t(y)) = 0$ which we can solve.
A necesary and sufficient condition for $2\alpha(y^2 - \frac{19}{4}) + \alpha^2 + 11y + \frac{139}{4}$ to be a perfect square is that the discriminant of the quadratic in y is 0. Writing the quadratic out we get:

$2\alpha y^2 + 11y + (\alpha^2 - \frac{19\alpha}{2} + \frac{139}{4})$ (2.7.12)
The discriminant of (2.7.12) is:
$\Delta = 121 - 8\alpha(\alpha^2 - \frac{19\alpha}{2} + \frac{139}{4})$ (2.7.13)

Thus the condition $\Delta = 0$ becomes:
$\alpha^3 - \frac{19\alpha^2}{2} + \frac{139\alpha}{4} - \frac{121}{8} = 0$ (2.7.14)

But we can solve (2.7.14) because it is a cubic - just follow the approach set out in my cubics paper:
Let $\alpha = \beta + f$ (2.7.15)
and substitute into (2.7.14), expand and collect like terms:

$\beta^3 + (3f - \frac{19}{2})\beta^2 + (3f^2 - 19f + \frac{139}{4})\beta + (f^3 - \frac{19}{2}f^2 + \frac{139}{4}f - \frac{121}{8}) = 0$ (2.7.16)

As before we want the square term in β to go so:

$3f - \frac{19}{2} = 0 \implies f = \frac{19}{6}$ (2.7.17)

Equation (2.7.16) then becomes:

$\beta^3 + P\beta + Q = 0$ (2.7.18) where:

$P = 3f^2 - 19f + \frac{139}{4}$ (2.17.19)
$Q = f^3 - \frac{19}{2}f^2 + \frac{139}{4}f - \frac{121}{8}$ (2.7.20)
Substituting $f = \frac{19}{6}$ into (2.7.19) and (2.7.20) we get:

$P = \frac{14}{3}$ (2.7.21)

$Q = \frac{848}{27}$ (2.7.22)

Thus (2.7.18) becomes:
$\beta^3 + \frac{14}{3}\beta + \frac{848}{27} = 0$ (2.7.23)

To solve (2.7.23) we let $\beta = \mu + \nu$ (2.7.24)
and substitute into (2.7.23), expand and collect like terms:

$\mu^3 + \nu^3 + (\mu + \nu)(\frac{14}{3} + 3\mu\nu) + \frac{848}{27} = 0$ (2.7.25)
Now we need to make: $\frac{14}{3} + 3\mu\nu = 0 \implies \mu\nu = -\frac{14}{9}$ (2.7.26)
Hence when we interpret μ^3 and ν^3 as roots we have the following relationships:

$\mu^3 \nu^3 = -(\frac{14}{9})^3$ (2.7.27)
$\mu^3 + \nu^3 = -\frac{848}{27}$ (2.7.28)

Using Viète's theorem for the roots μ^3 and ν^3 we have that those roots satisfy:
$w^2 + \frac{848}{27}w - (\frac{14}{9})^3 = 0$ (2.7.29)

Now it gets very messy from here on in so rather than obscuring what is going on with lots of square and cube roots I will resort to numerical answers via Mathematica.
Using Mathematica's Solve/NSolve we get that the roots of (2.7.29) are:

$w_1 = \frac{2}{27}(-212 - 39\sqrt{30} = -31.5268 = \mu^3$ (2.7.30)
$w_1 = \frac{2}{27}(-212 + 39\sqrt{30} = 0.11939 = \nu^3$ (2.7.31)
To find the three roots of each of μ and ν we have to solve the following cubics:

$\mu^3 + 31.5268 = 0$ (2.7.32)
$\nu^3 - 0.11939 = 0$ (2.7.33)
Solving (2.7.32) and (2.7.33) we have:
$\mu_1 = -3.15908$ (2.7.34)
$\mu_2 = 1.57954 - 2.73584i$ (2.7.35)
$\mu_3 = 1.57954 + 2.73584i$ (2.7.36)

$\nu_1 = -0.246204 - 0.426438i$ (2.7.37)
$\nu_2 = -0.246204 + 0.426438i$ (2.7.38)
$\nu_3 = 0.492409$ (2.7.39)

Working back to α we need to use (2.7.15),(2.7.17) and (2.7.24) so that:

$\alpha = \beta + f = \mu + \nu + \frac{19}{6}$ (2.7.40) where we take into account the roots of μ and ν

We find that:

$\mu_1 + \nu_1 + \frac{19}{6} == -0.238613 - 0.426438i$ (2.7.41)

$\mu_1 + \nu_2 + \frac{19}{6} == -0.238613 + 0.426438i$ (2.7.42)

$\mu_1 + \nu_3 + \frac{19}{6} = \frac{1}{2}$ (2.7.43)

$\mu_2 + \nu_1 + \frac{19}{6} = 4.5 - 3.16228i$ (2.7.44)

$\mu_2 + \nu_2 + \frac{19}{6} = 4.5 - 2.3094i$ (2.7.45)

$\mu_2 + \nu_3 + \frac{19}{6} = 5.23861 - 2.73584i$ (2.7.46)

$\mu_3 + \nu_1 + \frac{19}{6} = 4.5 + 2.3094i \quad (2.7.47)$

$\mu_3 + \nu_2 + \frac{19}{6} = 4.5 + 3.16228i \quad (2.7.48)$

$\mu_3 + \nu_3 + \frac{19}{6} = 5.23861 + 2.73584i \quad (2.7.49)$

Note that if we solve (2.7.14) using Solve in Mathematica we get the three roots as:

$\alpha_1 = \frac{1}{2}$ corresponds to (2.7.43)
$\alpha_2 = \frac{1}{2}(9 - 2i\sqrt{10})$ corresponds to (2.7.44)
$\alpha_2 = \frac{1}{2}(9 + 2i\sqrt{10})$ corresponds to (2.7.48)

Because we need $2\alpha(y^2 - \frac{19}{4}) + \alpha^2 + 11y + \frac{139}{4}$ to be a perfect square why not try $\alpha = \frac{1}{2}$?

We get:

$2 \times \frac{1}{2}(y^2 - \frac{19}{4}) + \frac{1}{4} + 11y + \frac{139}{4} = y^2 + 11y + \frac{121}{4} = (y + \frac{11}{2})^2 \quad (2.7.49)$
Thus $\alpha = \frac{1}{2}$ does indeed generate the required perfect square and that is all we need to solve (2.7.11) which is done as follows:

$(y^2 - \frac{19}{4} + \alpha)^2 - [2\alpha(y^2 - \frac{19}{4}) + \alpha^2 + 11y + \frac{139}{4}] = 0$ and on substituting $\alpha = \frac{1}{2}$:
$(y^2 - \frac{17}{4})^2 - (y + \frac{11}{2})^2 = 0$
$\left[(y^2 - \frac{17}{4}) - (y + \frac{11}{2})\right]\left[(y^2 - \frac{17}{4}) + (y + \frac{11}{2}\right] = 0$

Hence we have to solve the following two quadratics:

$y^2 - y - \frac{39}{4} = 0 \quad (2.7.50)$
$y^2 + y + \frac{5}{4} = 0 \quad (2.7.51)$

The solutions to (2.7.50) are: $y = \frac{1}{2} \pm \sqrt{10} \quad (2.7.52)$
The solutions to (2.7.51) are: $y = -\frac{1}{2} \pm i \quad (2.7.53)$

Recalling from (2.7.3) and (2.7.5) that $x = y + \frac{3}{2}$ we have that:

$$x_{1,2,3,4} = \begin{cases} \frac{1}{2} + \sqrt{10} + \frac{3}{2} = 2 + \sqrt{10} \\ \frac{1}{2} - \sqrt{10} + \frac{3}{2} = 2 - \sqrt{10} \\ -\frac{1}{2} + i + \frac{3}{2} = 1 + i \\ -\frac{1}{2} - i + \frac{3}{2} = 1 - i \end{cases} \quad (2.7.54)$$

Going back to (2.7.2) these roots are what Mathematica gave us - without all the tedium!!

The reason I chose $\alpha = \frac{1}{2}$ to generate the perfect square is because, out of the nine possibilities in (2.7.41)-(2.7.49), it was the only one which gave the required perfect square.

Clearly this is a fiddly process and one has to have some admiration for Cardano and his students in working doing this stuff without computers. It also explains why in high school the solutions to cubics and quartics are not usually pursued.