Some cheeky Mathematical Tripos Part 1A exam problems

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1 Background

In the May 2019 Part 1A Cambridge Mathematical Tripos examination a couple of problems caught my eye for being like a "cheeky" white wine - inviting you to quaff them quickly in a mathematical sense. See what you think.

As usual there are plenty of problems that require proficiency in technique to get through them in the space of 3 hours.

2 Problem 1C(a)

If

$$x + iy = \sum_{a=0}^{200} i^a + \prod_{b=1}^{50} i^b \tag{1}$$

where $x, y \in \mathbb{R}$, what is the value of xy ?

2.1 Solution

Once you write $i^a = e^{\frac{i\pi a}{2}}$ and $i^b = e^{\frac{i\pi b}{2}}$ it is all over provided you resist the temptation to sum a geometric series and simply draw the graphs of $\sin x$ and $\cos x$ and all will be revealed:

$$\sum_{a=0}^{200} i^a = \sum_{a=0}^{200} e^{\frac{i\pi a}{2}} = 1 \tag{2}$$

To see this note the symmetries involved here. For $a = 0, 1, 2, \dots, 200$, $\cos\frac{\pi a}{2} = 0$ if a is odd and ± 1 otherwise. There are 201 terms in the sum of which 100 odd terms are zero and the remaining 100 even terms pairwise cancel eg $\cos\frac{2\pi}{2} = -1$ and $\cos\frac{4\pi}{2} = +1$. This leaves as the only contributing term in the sum coming from $a = 0$ ie $\cos 0 = 1$. So $x = 1$ (as you will see there is no real component in the second term).

Now for the other term.

$$\begin{aligned}\prod_{b=1}^{50} i^b =& \prod_{b=1}^{50} e^{\frac{i\pi b}{2}} \\ =& e^{\frac{i\pi \sum_{b=1}^{50} b}{2}} \\ =& e^{\frac{i\pi 25\times 51}{2}} \\ =& e^{\frac{i\pi 1275}{2}}\end{aligned} \tag{3}$$

We already know that $\cos\frac{1275\pi}{2} = 0$ so we are left with working out $\sin\frac{1275\pi}{2}$ to get the imaginary part of (3). Clearly for $k = 0, 1, \dots$ (just look at a graph):

$$\sin\left(\frac{\pi}{2} + 2k\pi\right) = 1 \tag{4}$$

and

$$\sin\left(\frac{3\pi}{2} + 2k\pi\right) = -1 \tag{5}$$

There must be an integral k that solves (4) or (5). In (4) we would need $4k + 1 = 1275$ but this would mean that $k = 318.5$ and so that is wrong, but from (5), $4k + 3 = 1275$ works since $k = 318$. Hence $\sin\frac{1275\pi}{2} = -1$ and so $y = -1$.

In conclusion, $xy = -1$.

3 Problem 1C(b)

Evaluate

$$\frac{(1+i)^{2019}}{(1-i)^{2017}} \tag{6}$$

3.1 Solution

To solve this all we have to do is note (draw a diagram and it all becomes clear) that:

$$1 + i = \sqrt{2}e^{\frac{\pi i}{4}} \tag{7}$$

and

$$1 - i = \sqrt{2}e^{\frac{-\pi i}{4}} \tag{8}$$

So all we do now is some high school stuff:

$$\begin{aligned}\frac{(1+i)^{2019}}{(1-i)^{2017}} =& \frac{(\sqrt{2}e^{\frac{\pi i}{4}})^{2019}}{(\sqrt{2}e^{\frac{-\pi i}{4}})^{2017}} \\ =& 2e^{1009\pi i} \\ =& -2\end{aligned} \tag{9}$$

The last time I tuned in, $e^{1009\pi i} = 1$ since $\sin(2k+1)\pi = 0$ for $k = 0, 1, 2, \ldots$ and $\cos(2k+1)\pi = -1$ for $k = 0, 1, 2, \ldots$. Doubt me - draw a graph.

That was a really cheeky little problem!

4 Problem 1C(c)

Find a complex number z such that:

$$i^{i^z} = 2 \tag{10}$$

You just know they are playing with you with this type of problem. This is the Crocodile Dundee exponential - that's not an exponential, THIS is an exponential!

In what follows $\log z$ is the logarithm to the base e. Recall that for any two complex numbers z and w:

$$z^w = e^{w \log z} = e^{w[\log|z| + i(\theta + 2n\pi)]} \tag{11}$$

The "principal part" of $\log z$ is Log $z = \log|z| + i\theta$ where $-\pi < \theta \leq \pi$.
In this "tower" we can write:

$$\begin{aligned} w = i^z =& e^{z(\log|i| + \frac{i\pi}{2})} \\ =& e^{\frac{i\pi z}{2}}\end{aligned} \tag{12}$$

Hence:

$$
\begin{aligned}
i^{i^z} = i^w =& e^{\frac{i\pi w}{2}} \\
=& e^{\frac{i\pi i^z}{2}} \\
=& e^{\frac{\pi i^{z+1}}{2}} \\
=& 2
\end{aligned}
\tag{13}
$$

Therefore:

$$
\begin{aligned}
\log e^{\frac{\pi i^{z+1}}{2}} =& \log 2 \\
\frac{\pi i^{z+1}}{2} =& \log 2 \\
i^{z+1} =& \frac{2}{\pi}\log 2 \\
e^{\frac{i\pi(z+1)}{2}} =& \frac{2}{\pi}\log 2 \\
\log\left(e^{\frac{i\pi(z+1)}{2}}\right) =& \log\left(\frac{2}{\pi}\log 2\right) \\
\frac{i\pi(z+1)}{2} =& \log\left(\frac{2}{\pi}\log 2\right) \\
z =& \frac{2}{i\pi}\log\left(\frac{2}{\pi}\log 2\right) - 1
\end{aligned}
\tag{14}
$$

Thankfully Mathematica validates this answer!

In[10]:= $z = \frac{2}{\mathbb{i}\,\pi}\ \text{Log}\left[\ \frac{2}{\pi}\ \text{Log}[2]\right]\ - 1;$

In[11]:= $\mathbb{i}^{\mathbb{i}^{z}}$

Out[11]= 2

5 Problem 1C(d)

Interpret geometrically the curve defined by the set of points satisfying:

$$\log z = i \log \bar{z} \tag{15}$$

in the complex plane.

5.1 Solution

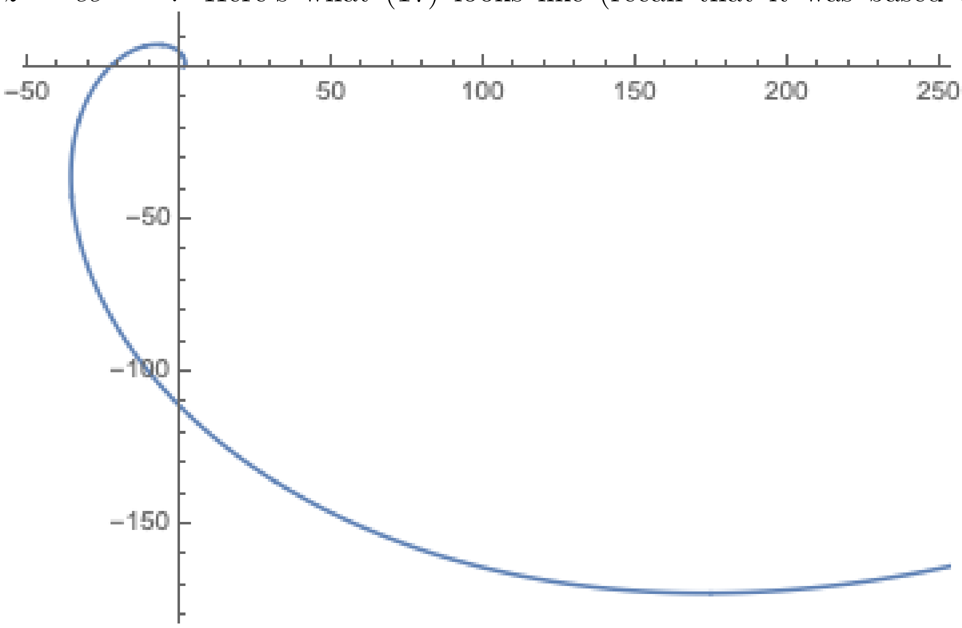
Using the concepts of Problem 1C(c) we have the following from (15):

$$\begin{aligned}
\log z =& \log|z| + i(\theta + 2n\pi i) \\
=& i\big(\log|\bar{z}| - i(\theta + 2n\pi)\big) \\
=& i\log|z| + \theta + 2n\pi \\
(1-i)\log|z| =& (\theta + 2n\pi)(1-i) \\
\log|z| =& \theta + 2n\pi \\
\therefore |z| =& e^{\theta + 2n\pi}
\end{aligned} \tag{16}$$

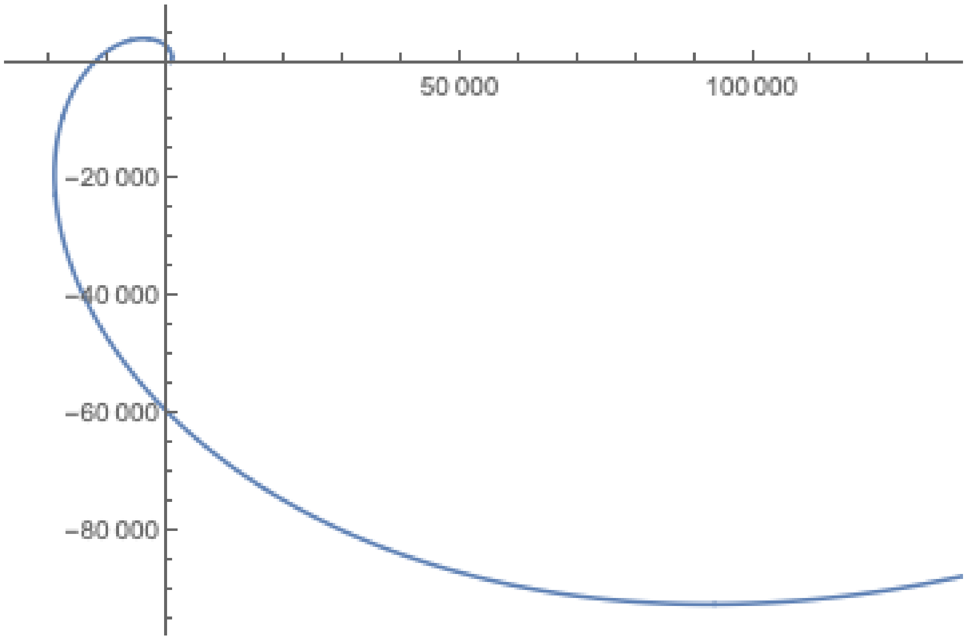
This will give rise to a family of curves.: $z = e^{\theta+2n\pi}e^{i\theta}$. Let us take $n = 0$. Then since $z = |z|e^{i\theta}$ we have that:

$$z = e^{\theta}\, e^{i\theta} = e^{\theta(1+i)} \tag{17}$$

This looks suspiciously like a logarithmic spiral (who would have thought with all those logs flying around!). The general form of a logarithmic spiral is $z = ce^{(k+i)\theta}$. Here's what (17) looks like (recall that it was based on $n = 0$):



Here is the spiral for $n = 1$:



The family grows explosively thanks to the exponential radial factor.

6 Problem 3E

State the Bolzano-Weierstrass theorem.

Let (a_n) be a sequence of non-zero real numbers. Which of the following conditions is sufficient to ensure that $(\frac{1}{a_n})$ converges. Give a proof or counter-example as appropriate.

(i) $a_n \to l$ for some real number l
(ii) $a_n \to l$ for some non-zero real number l
(iii) (a_n) has no convergent subsequence.

6.1 Solution

(i) Counter-example: $a_n = \frac{1}{n} \to 0$ but $\frac{1}{a_n} = n$ diverges.

(ii) This one requires a proof which mimics the proof that if $a_n \to a$ and $b_n \to b$ then $\frac{a_n}{b_n} \to \frac{a}{b}$ when $b_n \neq 0 \quad \forall n$ and $b \neq 0$.

As usual we take $\epsilon > 0$. If (a_n) converges to $l \neq 0$ then $(|a_n|)$ converges to $|l| > 0$. Why? Just recall the inequality $|a - b| \geq \big||a| - |b|\big|$.

We have:

$\left|\frac{1}{a_n} - \frac{1}{l}\right| = \frac{|a_n - l|}{a_n|l|}$, noting that $a_n \neq 0$ for all n and $l \neq 0$.

We can find an N_1 such that $\big||a| - |b|\big| < \frac{|l|}{2}$ because of the convergence of $|a_n|$. Thus $\frac{|l|}{2} < a_n < \frac{3|l|}{2}$ and so $\frac{l^2}{2} < |l|a_n$. From the convergence of a_n we have that there exists an N_2 such that $|a_n - l| < \frac{l^2\epsilon}{2}$. So for $n > N = \max\{N_1, N_2\}$ we have:

$\left|\frac{1}{a_n} - \frac{1}{l}\right| = \frac{|a_n - l|}{a_n|l|} < \frac{l^2\epsilon}{2}\frac{2}{l^2} = \epsilon$ as required.

(iii) This part actually requires the Bolzano-Weierstrass theorem which states that every bounded sequence has a convergent subsequence.

If a_n is bounded then it has a convergent subsequence and the contrapositive of this is that if a_n has no convergent subsequence it must be unbounded. Hence you have a divergent sequence of positive terms ie they get arbitrarily big so $\frac{1}{a_n}$ must converge to zero.

7 References

Mathematical Tripos Part 1A, May 2019.

8 History

Created 08 November 2019