Some cheeky Mathematical Tripos Part 1A exam problems

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November 8, 2019

1 Background

In the May 2019 Part 1A Cambridge Mathematical Tripos examination a couple of problems caught my eye for being like a “cheeky” white wine - inviting you to quaff them quickly in a mathematical sense. See what you think.

As usual there are plenty of problems that require proficiency in technique to get through them in the space of 3 hours.

2 Problem 1C(a)

If

\[ x + iy = \sum_{a=0}^{200} i^a + \prod_{b=1}^{50} i^b \]  

(1)

where \( x, y \in \mathbb{R} \), what is the value of \( xy \)?

2.1 Solution

Once you write \( i^a = e^{i\pi a} \) and \( i^b = e^{i\pi b} \) it is all over provided you resist the temptation to sum a geometric series and simply draw the graphs of \( \sin x \) and \( \cos x \) and all will be revealed:

\[ \sum_{a=0}^{200} i^a = \sum_{a=0}^{200} e^{i\pi a} = 1 \]  

(2)
To see this note the symmetries involved here. For \( a = 0, 1, 2, \ldots, 200 \), \( \cos \frac{2a\pi}{2} = 0 \) if \( a \) is odd and \( \pm 1 \) otherwise. There are 201 terms in the sum of which 100 odd terms are zero and the remaining 100 even terms pairwise cancel eg \( \cos \frac{2\pi}{2} = -1 \) and \( \cos \frac{4\pi}{2} = +1 \). This leaves as the only contributing term in the sum coming from \( a = 0 \) ie \( \cos 0 = 1 \). So \( x = 1 \) (as you will see there is no real component in the second term).

Now for the other term.

\[
\prod_{b=1}^{50} i^b = \prod_{b=1}^{50} e^{\frac{ib\pi}{2}} \\
= e^{\frac{i\sum_{b=1}^{50} b}{2}} \\
= e^{\frac{i\sum_{b=51}^{100} b}{2}} \\
= e^{i\frac{1275}{2}}
\]

We already know that \( \cos \frac{1275\pi}{2} = 0 \) so we are left with working out \( \sin \frac{1275\pi}{2} \) to get the imaginary part of (3). Clearly for \( k = 0, 1, \ldots \) (just look at a graph):

\[
\sin \left( \frac{\pi}{2} + 2k\pi \right) = 1 \quad (4)
\]

and

\[
\sin \left( \frac{3\pi}{2} + 2k\pi \right) = -1 \quad (5)
\]

There must be an integral \( k \) that solves (4) or (5). In (4) we would need \( 4k + 1 = 1275 \) but this would mean that \( k = 318.5 \) and so that is wrong, but from (5), \( 4k + 3 = 1275 \) works since \( k = 318 \). Hence \( \sin \frac{1275\pi}{2} = -1 \) and so \( y = -1 \).

In conclusion, \( xy = -1 \).

3 Problem 1C(b)

Evaluate

\[
\frac{(1 + i)^{2019}}{(1 - i)^{2017}}
\]
3.1 Solution

To solve this all we have to do is note (draw a diagram and it all becomes clear) that:

\[ 1 + i = \sqrt{2} e^{\frac{i\pi}{4}} \quad (7) \]

and

\[ 1 - i = \sqrt{2} e^{-\frac{\pi i}{4}} \quad (8) \]

So all we do now is some high school stuff:

\[
\frac{(1 + i)^{2019}}{(1 - i)^{2017}} = \frac{(\sqrt{2} e^{\frac{i\pi}{4}})^{2019}}{(\sqrt{2} e^{-\frac{\pi i}{4}})^{2017}} = 2 e^{1009\pi i} = -2 \quad (9)
\]

The last time I tuned in, \( e^{1009\pi i} = 1 \) since \( \sin(2k + 1)\pi = 0 \) for \( k = 0, 1, 2, \ldots \) and \( \cos(2k + 1)\pi = -1 \) for \( k = 0, 1, 2, \ldots \). Doubt me - draw a graph.

That was a really cheeky little problem!

4 Problem 1C(c)

Find a complex number \( z \) such that:

\[ i^z = 2 \quad (10) \]

You just know they are playing with you with this type of problem. This is the Crocodile Dundee exponential - that’s not an exponential, THIS is an exponential!

In what follows \( \log z \) is the logarithm to the base \( e \). Recall that for any two complex numbers \( z \) and \( w \):

\[ z^w = e^{w \log z} = e^{w[\log |z| + i(\theta + 2n\pi)]} \quad (11) \]

The ”principal part” of \( \log z \) is \( \text{Log } z = \log |z| + i\theta \) where \(-\pi < \theta \leq \pi\). In this ”tower” we can write:

\[
w = i^z = e^{i(\log |z| + \frac{i\pi}{2})} = e^{\frac{iz^2}{2}} \quad (12)
\]
Hence:

\[ i^i = i^w = e^{\frac{i\pi w}{2}} = e^{\frac{i\pi i^2}{2}} = e^{\frac{i\pi (i+1)}{2}} = 2 \]  

Therefore:

\[
\log e^{\frac{i\pi (z+1)}{2}} = \log 2 \\
\frac{i\pi (z+1)}{2} = \log 2 \\
iz + 1 = \frac{2}{\pi} \log 2 \\
e^{\frac{i\pi (z+1)}{2}} = \frac{2}{\pi} \log 2 \\
\log \left( e^{\frac{i\pi (z+1)}{2}} \right) = \log \left( \frac{2}{\pi} \log 2 \right) \\
\frac{i\pi (z+1)}{2} = \log \left( \frac{2}{\pi} \log 2 \right) \\
z = \frac{2}{i\pi} \log \left( \frac{2}{\pi} \log 2 \right) - 1
\]

Thankfully Mathematica validates this answer!

\[
\text{In[10]} := \quad z = \frac{2}{i\pi} \log \left[ \frac{2}{\pi} \log[2] \right] - 1; \\
\text{In[11]} := \quad i^i z \\
\text{Out[11]} := \quad 2
\]

5 Problem 1C(d)

Interpret geometrically the curve defined by the set of points satisfying:

\[ \log z = i \log \bar{z} \]

in the complex plane.
5.1 Solution

Using the concepts of Problem 1C(c) we have the following from (15):

\[
\begin{align*}
\log z &= \log |z| + i(\theta + 2n\pi i) \\
&= i(\log |z| - i(\theta + 2n\pi)) \\
&= i \log |z| + \theta + 2n\pi \\
(1 - i) \log |z| &= (\theta + 2n\pi)(1 - i) \\
\log |z| &= \theta + 2n\pi \\
\therefore |z| &= e^{\theta + 2n\pi}
\end{align*}
\]

This will give rise to a family of curves: \( z = e^{\theta + 2n\pi} e^{i\phi} \). Let us take \( n = 0 \). Then since \( z = |z| e^{i\phi} \) we have that:

\[
z = e^{\theta} e^{i\phi} = e^{\theta(1+i)}
\]

This looks suspiciously like a logarithmic spiral (who would have thought with all those logs flying around!). The general form of a logarithmic spiral is \( z = ce^{(k+i)\theta} \). Here’s what (17) looks like (recall that it was based on \( n = 0 \)):

Here is the spiral for \( n = 1 \):
6 Problem 3E

State the Bolzano-Weierstrass theorem.

Let \((a_n)\) be a sequence of non-zero real numbers. Which of the following conditions is sufficient to ensure that \((\frac{1}{a_n})\) converges. Give a proof or counter-example as appropriate.

(i) \(a_n \to l\) for some real number \(l\)
(ii) \(a_n \to l\) for some non-zero real number \(l\)
(iii) \((a_n)\) has no convergent subsequence.

6.1 Solution

(i) Counter-example: \(a_n = \frac{1}{n} \to 0\) but \(\frac{1}{a_n} = n\) diverges.

(ii) This one requires a proof which mimics the proof that if \(a_n \to a\) and \(b_n \to b\) then \(\frac{a_n}{b_n} \to \frac{a}{b}\) when \(b_n \neq 0\) \(\forall n\) and \(b \neq 0\).

As usual we take \(\epsilon > 0\). If \((a_n)\) converges to \(l \neq 0\) then \((|a_n|)\) converges to \(|l| > 0\). Why? Just recall the inequality \(|a - b| \geq |a| - |b|\).
We have:

\[ |\frac{1}{a_n} - \frac{1}{l}| = \frac{|a_n - l|}{a_n|l|}, \text{ noting that } a_n \neq 0 \text{ for all } n \text{ and } l \neq 0. \]

We can find an \( N_1 \) such that \( ||a| - |b|| < |l|/2 \) because of the convergence of \( |a_n| \).

Thus \( \frac{|l|}{2} < a_n < \frac{3|l|}{2} \) and so \( \frac{l^2}{2} < |l|a_n \). From the convergence of \( a_n \) we have that there exists an \( N_2 \) such that \( |a_n - l| < \frac{l^2}{2} \). So for \( n > N = \max\{N_1, N_2\} \) we have:

\[ |\frac{1}{a_n} - \frac{1}{l}| = \frac{|a_n - l|}{a_n|l|} < \frac{l^2}{2 \cdot \frac{l^2}{2}} = \epsilon \text{ as required.} \]

(iii) This part actually requires the Bolzano-Weierstrass theorem which states that every bounded sequence has a convergent subsequence.

If \( a_n \) is bounded then it has a convergent subsequence and the contrapositive of this is that if \( a_n \) has no convergent subsequence it must be unbounded. Hence you have a divergent sequence of positive terms i.e. they get arbitrarily big so \( \frac{1}{a_n} \) must converge to zero.

7 References


8 History

Created 08 November 2019