Spherical Bessel functions in quantum mechanics

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June 10, 2022

1 Introduction

Bessel functions are usually introduced in undergraduate Fourier analysis or engineering courses in the context of hanging chains and vibrating circular membranes, for example. Bessel functions and Fourier transforms figured prominently in the discovery of the double helix structure of DNA via X-ray diffraction. However, the integral form of Bessel's function actually arose from Bessel's analysis of the eccentric anomaly in elliptic planetary motion - see [1] pages 4-5. His integral form looked like this:

$$A_r = \frac{2}{r\pi}\int_0^{\pi} \cos[r\phi - e\sin\phi)]\,d\phi \tag{1}$$

Here e is the eccentricity of the ellipse and in what follows $x = e$ so that (1) is $A_r = \frac{2}{r\pi}\int_0^{\pi} \cos[r\phi - x\sin\phi)]\,d\phi$. It can be shown (see [1], page 5) that with $u = \frac{\pi r}{2}A_r$ that u satisfies Bessel's equation:

$$\frac{d^2u}{dx^2} + \frac{1}{x}\frac{du}{dx} + (r^2 - \frac{r^2}{x^2})u = 0 \tag{2}$$

If the substitution $z = rx$ is made in (2) you get what is regarded as the standard form of Bessel's equation:

$$\frac{d^2u}{dz^2} + \frac{1}{z}\frac{du}{dz} + (1 - \frac{r^2}{z^2})u = 0 \tag{3}$$

In polar coordinates, the Schrodinger wave equation for a free particle leads for each value l of the orbital angular momentum to a radial equation of the following form:

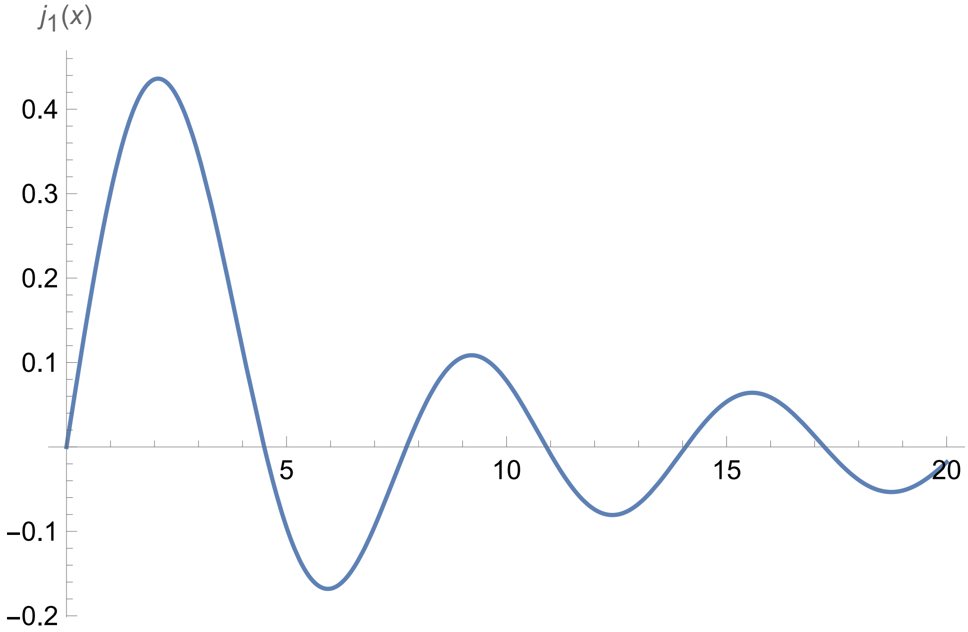
$$[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + 1 - \frac{l(l+1)}{r^2}]f_l = 0 \tag{4}$$

See [2], pages 488-489 and [3], pages 938-947 .

Messiah ([2], page 488-89) says that spherical Bessel functions:

$$j_l(r) = \left(\frac{\pi}{2r}\right)^{\frac{1}{2}} J_{l+\frac{1}{2}}(r) \tag{5}$$

solve the radial equation (4). The $J_{l+\frac{1}{2}}(r)$ are the usual Bessel functions of order $l + \frac{1}{2}$ where $l = 0, 1, 2, 3, \ldots$. The purpose of this short note is to prove this. It is a minor homework example in such courses. $j_1(x)$ looks like this:



2 Proving that spherical Bessel functions solve the radial equation

To prove the result all we need to do is to differentiate $j_l(r)$ as in (4) and use the fact that $u = J_{l+\frac{1}{2}}(r)$ satisfies Bessel's equation and then do some algebbraic manipulations ie:

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} + \left(1 - \frac{(l+\frac{1}{2})^2}{r^2}\right)u = 0 \tag{6}$$

Churchill deals with the properties of $J_\nu(x)$ for real ν in terms of satisfying (6) in [4] at pages 164-169.

Differentiating we have:

$$\frac{du}{dr} = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \left\{ r^{-\frac{1}{2}} J'_{l+\frac{1}{2}}(r) - \frac{1}{2} r^{-\frac{3}{2}} J_{l+\frac{1}{2}}(r) \right\} \tag{7}$$

and

$$\frac{d^2u}{dr^2} = \Big(\frac{2}{\pi}\Big)^{\frac{1}{2}}\Big\{r^{-\frac{1}{2}}J''_{l+\frac{1}{2}}(r) - r^{-\frac{3}{2}}J'_{l+\frac{1}{2}}(r) + \frac{3}{4}r^{-\frac{5}{2}}J_{l+\frac{1}{2}}(r)\Big\} \tag{8}$$

Thus (4) becomes:

$$\begin{aligned}
\frac{d^2u}{dr^2} + \frac{2}{r}\frac{du}{dr} + \Big(1 - \frac{l(l+1)}{r^2}\Big)u =& \Big(\frac{2}{\pi}\Big)^{\frac{1}{2}}\Big\{r^{-\frac{1}{2}}J''_{l+\frac{1}{2}}(r) - r^{-\frac{3}{2}}J'_{l+\frac{1}{2}}(r) \\
&+ \frac{3}{4}r^{-\frac{5}{2}}J_{l+\frac{1}{2}}(r)\Big\} + \frac{2}{r}\Big(\frac{2}{\pi}\Big)^{\frac{1}{2}}\Big\{r^{-\frac{1}{2}}J'_{l+\frac{1}{2}}(r) - \frac{1}{2}r^{-\frac{3}{2}}J_{l+\frac{1}{2}}(r)\Big\} \\
&+ \Big(\frac{2}{\pi}\Big)^{\frac{1}{2}}\Big(1 - \frac{l(l+1)}{r^2}\Big)r^{-\frac{1}{2}}J_{l+\frac{1}{2}}(r) \\
=& \Big(\frac{2}{\pi}\Big)^{\frac{1}{2}}\Big[r^{-\frac{1}{2}}J''_{l+\frac{1}{2}}(r) + r^{-\frac{3}{2}}J'_{l+\frac{1}{2}}(r) + \Big(-\frac{1}{4}r^{-\frac{5}{2}} \\
&+ r^{-\frac{1}{2}} - r^{-\frac{1}{2}}\frac{l(l+1)}{r^2}\Big)J_{l+\frac{1}{2}}(r)\Big] \\
=& \Big(\frac{2}{\pi}\Big)^{\frac{1}{2}}r^{-\frac{1}{2}}\Big[J''_{l+\frac{1}{2}}(r) + \frac{1}{r}J'_{l+\frac{1}{2}}(r) + \Big(-\frac{1}{4}r^{-2} + 1 - \frac{l(l+1)}{r^2}\Big)J_{l+\frac{1}{2}}(r)\Big] \\
=& \Big(\frac{2}{\pi}\Big)^{\frac{1}{2}}r^{-\frac{1}{2}}\Big[\underbrace{J''_{l+\frac{1}{2}}(r) + \frac{1}{r}J'_{l+\frac{1}{2}}(r) + \Big(1 - \frac{(l+\frac{1}{2})^2}{r^2}\Big)J_{l+\frac{1}{2}}(r)}_{=0} \\
&+ \Big(\frac{(l+\frac{1}{2})^2}{r^2} - \frac{1}{4}r^{-2} - \frac{l(l+1)}{r^2}\Big)J_{l+\frac{1}{2}}(r)\Big] \\
=& \Big(\frac{2}{\pi}\Big)^{\frac{1}{2}}r^{-\frac{1}{2}}\Big[\frac{(l+\frac{1}{2})^2}{r^2} - \frac{1}{4}r^{-2} - \frac{l(l+1)}{r^2}\Big]J_{l+\frac{1}{2}}(r) \\
=& \Big(\frac{2}{\pi}\Big)^{\frac{1}{2}}r^{-\frac{1}{2}}\Big[\frac{l^2+l+\frac{1}{4}}{r^2} - \frac{1}{4r^2} - \frac{(l^2+l)}{r^2}\Big]J_{l+\frac{1}{2}}(r) \\
=& \Big(\frac{2}{\pi}\Big)^{\frac{1}{2}}r^{-\frac{1}{2}}\Big[\frac{1}{4}r^2 - \frac{1}{4}r^2\Big]J_{l+\frac{1}{2}}(r) \\
=& 0
\end{aligned} \tag{9}$$

3 References

[1] Andrew Gray, G B Mathews, , A Treatise on Bessel Functions and their Applications to Physics, Merchant Books, 2007

[2] Albert Messiah, Quantum Mechanics , Two Volumes bound as One , Dover, 1999

[3] Claude Cohen-Tannoudji, Bernard Diu, Franck Laloë, Quantum Mechanics , Volume 2, John Wiley and Sons, 1977.

[4] Ruel V Churchill Fourier Series and Boundary Value Problems, Second Edition, McGraw-Hill, 1963

4 History

Created 10 June 2022