Sums of normal variables - Lagrange's limiting approach

Background

George Marsaglia's paper

George Marsaglia' s paper on this subject (A*merican Statistical Association Journal,* March 1965 pp 193 - 205) contains a number of intermediate results leading up to the main result which involves the ratio of independent uniform random variables. He derives the distribution of $\sum_{k=1}^{n} c_k u_k$ where the u_k are independent and uniform on (0,1) and the $c_k > 0$ ∀k. When the $c_k = 1$ ∀k he comes up with the following formula for the probability that the sum is less than a fixed number:

$$P\{u_1 + \ldots + u_n < a\} = \frac{1}{n!} \sum_{i=0}^{\lceil a \rceil} (-1)^i (a-i)^n \tag{1}$$

Note that ⌈a⌉ is the greatest integer ≤ a.

Marsaglia' s proof of (1) is essentially inductive but obscures the inspiration for the induction. William Feller (" *An Introduction to Probability Theory and Its Applications*", Third Edition, Volume 1, Wiley, 1968, pp 284 - 295) presents three problems which provide the derivation of (1) as the limit of an infinite series. The derivation is essentially based on work by De Moivre and Lagrange. The three problems are posed below and solved.

Feller: Volume 1 Problem 18 page 284

Let $S_n = \sum_{k=1}^{n} X_k$ be the sum of mutually independent variables each assuming the values 1, 2,..., a with probability $\frac{1}{a}$. Show that the generating function is given by $P(s) = \left(\frac{s(1-s^a)}{a(1-s)}\right)^n$ and that for $j \geq n$:

$$P\{S_n = j\} = a^{-n} \sum_{v=0}^{\infty} (-1)^{v+j-n-av} \binom{n}{v} \binom{-n}{j-n-av}$$

$$= a^{-n} \sum_{v=0}^{\infty} (-1)^{v} \binom{n}{v} \binom{j-av-1}{n-1} \qquad (2)$$

Note that only finitely many terms in the sum are different from zero. For a = 6 you get the probability of scoring the sum j + n in a throw with a dice and the solution goes back to De Moivre.

The probability generating function (pgf) of one uniform variable with the defined characteristics is $\frac{(s+s^2+...+s^n)}{a}$ and for n such variables the pgf is the n-fold convolution, namely:

$P(s) = \left(\frac{(s+s^2+...+s^a)}{a}\right)^n$ which simplifies as follows:

$$P(s) = \left(\frac{s(1+s+...+s^{a-1})}{a}\right)^n$$

$$= \left(\frac{s(1-s^a)}{a(1-s)}\right)^n \qquad (3)$$

The required probability is the coefficient of s^j in (3).

Now $P(s) = a^{-n} s^n (1-s^a)^n (1-s)^{-n}$ where $-1 < s < 1$

$$= a^{-n} s^n \left(\sum_{v=0}^{\infty} \binom{n}{v} (-s^a)^v\right) \left(\sum_{m=0}^{\infty} \binom{-n}{m} (-s)^m\right) \qquad (4)$$

Note that $(1-s^a)^n$ can be written as an infinite series since all the factorials $\binom{n}{v}$ vanish when v > n ie there are finitely many terms. On the other hand $(1-s)^{-n}$ is an infinite series since $-n < 0$.

Equation (4) can be re-written as:

$$P(s) = a^{-n} \sum_{v=0}^{\infty} (-1)^{m+v} \binom{n}{v} \binom{-n}{m} (-s^a)^{av+m+n} \qquad (5)$$

Now let j = n + m + av and then j ≥ n for all n, m ≥ 0. Hence (5) becomes:

$$P(s) = a^{-n} \sum_{v=0}^{\infty} (-1)^{j-n-av+v} \binom{n}{v} \binom{-n}{j-n-av} s^j \qquad (6)$$

$$\text{Now} \quad \binom{-n}{m} = (-1)^m \binom{n+m-1}{m} \qquad (7)$$

Equation (7) can be established as follows assuming n > m ≥ 0 and noting that $\binom{n}{m} = \frac{n(n-1)\ldots(n-m+1)\,(n-m)!}{(n-m)!\,m!}$ there being m terms on the top.

$$\text{So} \binom{-n}{m} = \frac{-n\,(-n-1)\,(-n-2)\ldots(-n-m+1)}{m!}$$

$$= (-1)^m \frac{(n-1)!\,n(n+1)\,(n+2)\ldots(n+m-1)}{m!\,(n-1)!}$$

$$= (-1)^m \binom{n+m-1}{m} \qquad (8)$$

$$\text{Hence} \binom{-n}{j-n-av} = (-1)^{j-n-av} \binom{n+j-n-av-1}{j-n-av}$$

$$= (-1)^{j-n-av} \binom{j-av-1}{j-n-av}$$

But $\binom{k-1}{k-n} = \frac{(k-1)!}{(n-1)!\,(k-n)!} = \binom{k-1}{n-1}$ and letting k = j - av we get $\binom{j-av-1}{j-n-av} = \binom{j-av-1}{n-1}$

So (6) becomes $P(s) = a^{-n} \sum_{v=0}^{\infty} (-1)^{j-n-av+v} \binom{n}{v} (-1)^{j-n-av} \binom{j-av-1}{n-1} s^j$

$$= a^{-n} \sum_{v=0}^{\infty} (-1)^{2(j-n-av)+v} \binom{n}{v} \binom{j-av-1}{n-1} s^j$$

$$= a^{-n} \sum_{v=0}^{\infty} (-1)^{v} \binom{n}{v} \binom{j-av-1}{n-1} s^j$$

The required probability is the coefficient of s^j ie $a^{-n} \sum_{v=0}^{\infty} (-1)^v \binom{n}{v} \binom{j-av-1}{n-1}$

Feller: Volume 1 Problem 19 page 285

The next step is to show that the probability P{$S_n \leq$ j} has generating function $\frac{P(s)}{1-s}$ and hence P{$S_n \leq$ j} = $a^{-n} \sum_{v=0}^{\infty} (-1)^v \binom{n}{v} \binom{j-\text{av}}{n}$

Now P{$S_n \leq$ j} = P{$S_n =$ 1} + P{S_n= 2} +....+ P{S_n=j} and $\frac{P(s)}{1-s}$ = (1 + s + s^2 + s^3 + ...) P(s)

That is $\frac{P(s)}{1-s}$ = P(s) + s P(s) + s^2P(s) + ...

The required probability is the sum of the coefficients of powers of s in the expansion of $\frac{P(s)}{1-s}$.

Going back to P(s) = $a^{-n} s^n (1-s^a)^n (1-s)^{-n}$ we see that $\frac{P(s)}{1-s} = a^{-n} s^n (1-s^a)^n (1-s)^{-n} (1-s)^{-1} = a^{-n} s^n (1-s^a)^n (1-s)^{-n-1}$.

Hence $\frac{P(s)}{1-s} = a^{-n} s^n \left(\sum_{v=0}^{\infty} \binom{n}{v} (-s^a)^v\right) \left(\sum_{m=0}^{\infty} \binom{-n-1}{m} (-s)^m\right)$

$$= a^{-n} \sum_{v=0}^{\infty} (-1)^{m+v} \binom{n}{v} \binom{-n-1}{m} (-s^a)^{\text{av}+m+n}$$

Now let j = n + m + av and go through the same process as per the first problem and what you get is:

$$\frac{P(s)}{1-s} = a^{-n} \sum_{v=0}^{\infty} (-1)^{j-n-\text{av}+v} \binom{n}{v} \binom{-n-1}{j-n-\text{av}} s^j$$

$$= a^{-n} \sum_{v=0}^{\infty} (-1)^{j-n-\text{av}+v} \binom{n}{v} (-1)^{j-n-\text{av}} \binom{n+1+j-n-\text{av}-1}{j-n-\text{av}} s^j \qquad \text{using (8)}$$

$$= a^{-n} \sum_{v=0}^{\infty} (-1)^v \binom{n}{v} \binom{j-\text{av}}{j-n-\text{av}} s^j$$

$$= a^{-n} \sum_{v=0}^{\infty} (-1)^v \binom{n}{v} \binom{j-\text{av}}{n} s^j$$

Hence P{$S_n \leq$ j} = $a^{-n} \sum_{v=0}^{\infty} (-1)^v \binom{n}{v} \binom{j-\text{av}}{n}$ as required (9)

Feller: Volume 1 Problem 20 page 285

The final step is to find the limiting form of the probability as follows. If a → ∞ and j → ∞ so that $\frac{j}{a} \to$ x then P{$S_n \le$ j} → $\frac{1}{n!} \sum_{v=0}^{\infty} (-1)^v \binom{n}{v} (x - v)^n$ where the summation extends over all v such that 0 ≤ v < x

Starting with (9) we have:

$$P\{S_n \le j\} = a^{-n} \sum_{v=0}^{\infty} (-1)^v \binom{n}{v} \binom{j - av}{n}$$

$$= a^{-n} \sum_{v=0}^{\infty} (-1)^v \binom{n}{v} \frac{(j - av)!}{n!\,(j - av - n)!}$$

$$= a^{-n} \sum_{v=0}^{\infty} (-1)^v \binom{n}{v} \frac{(j - av)(j - av - 1)(j - av - 2)\dots(j - av - n + 1)(j - av - n)!}{n!\,(j - av - n)!}$$

$$= a^{-n} \sum_{v=0}^{\infty} (-1)^v \binom{n}{v} \frac{(j - av)(j - av - 1)(j - av - 2)\dots(j - av - n + 1)}{n!}$$

$$= \sum_{v=0}^{\infty} (-1)^v \binom{n}{v} \frac{\left(\frac{j}{a} - v\right)\left(\frac{j}{a} - v - \frac{1}{a}\right)\left(\frac{j}{a} - v - \frac{2}{a}\right)\dots\left(\frac{j}{a} - v - \frac{n}{a} + \frac{1}{a}\right)}{n!}$$ since there are n terms in the numerator

Hence P{$S_n \le$ j} → $\frac{1}{n!} \sum_{v=0}^{\lceil x \rceil} (-1)^v \binom{n}{v} (x - v)^n$ since each of the n terms in the numerator approaches (x -v) as a → ∞ and j → ∞ so that $\frac{j}{a} \to$ x

Feller notes that this result is due to Lagrange and in the theory of geometric probabilities the right hand side represents the distribution function of the sum of n independent random variables with "uniform" distribution in the interval (0,1)

There appears to be a minor typo in Marsaglia's paper in that he gives $P\{u_1 + \dots + u_n < a\}$ rather than $P\{u_1 + \dots + u_n \le a\}$ as derived above.