The Koide formula

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1 Background

Joan Baez recently drew attention to the Koide formula in his blog [1]. This remarkable empirical formula arose in 1981 from researches by Yoshio Koide. More detail on Koide's work can be found in [2].

There are three charged leptons: the electron, the muon and the tau. Let m_e, m_μ and m_τ be their masses. Then the Koide formula states that:

$$\frac{m_e + m_\mu + m_\tau}{(\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau})^2} = \frac{2}{3} \tag{1}$$

Suffice it to say that when you plug in the relevant experimentally obtained masses the formula is accurate to within the current experimental error bounds.

Quite amazing and something that gets the juices flowing to find a deeper explanation for this result. It may be nothing more than a numerical coincidence.

2 Bounds for the formula

It can be shown that whatever the masses of the electron, muon and tau may be the following inequality will hold for any positive values of m_e, m_μ and m_τ.

$$\frac{1}{3} \le \frac{m_e + m_\mu + m_\tau}{(\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau})^2} \le 1 \tag{2}$$

As I pointed out in the blog this result arises from the Cauchy-Schwarz inequality. Recall that the Cauchy-Schwarz inequality says that for $a_i > 0$ and $b_i > 0$:

$$\sum_{i=1}^{n} a_i b_i \le \Big(\sum_{i=1}^{n} a_i^2\Big)^{\frac{1}{2}} \Big(\sum_{i=1}^{n} b_i^2\Big)^{\frac{1}{2}} \tag{3}$$

Now if we let $b_i = 1$ for $i = 1, 2, 3$ we have:

$$\sqrt{a} + \sqrt{b} + \sqrt{c} \leq (a + b + c)^{\frac{1}{2}} \sqrt{3} \tag{4}$$

Squaring we get:

$$\frac{1}{3} \leq \frac{a + b + c}{(\sqrt{a} + \sqrt{b} + \sqrt{c})^2} \tag{5}$$

Now $(\sqrt{a} + \sqrt{b} + \sqrt{c})^2 > a + b + c$ hence $\frac{a+b+c}{\sqrt{a}+\sqrt{b}+\sqrt{c})^2} \leq 1$.

There is absolutely no physics in any of this!

Let's suppose we normalise the masses a, b, c . Our new masses become:

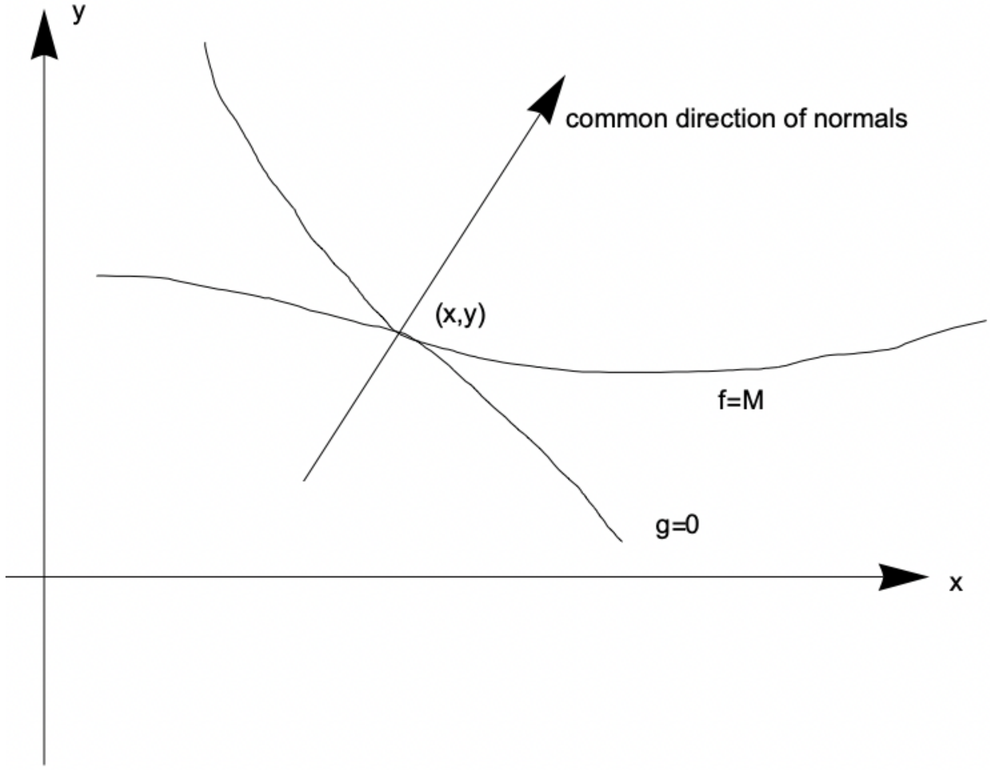
$\frac{a}{a+b+c}, \frac{b}{a+b+c}$ and $\frac{c}{a+b+c}$ which we denote by x, y, z respectively. Thus we have a constraint of the form:

$$g(x, y, z) = x + y + z - 1 = 0 \tag{6}$$

Now the thing we want to minimise subject to this constraint is:

$$f(x, y, z) = \frac{x + y + z}{(\sqrt{x} + \sqrt{y} + \sqrt{z})^2} \tag{7}$$

We recall that the general idea of Lagrange multipliers is embedded in this diagram:



The required relationship is:

$$\nabla f(x,y,z) = \lambda \nabla g(x,y,z) \tag{8}$$

subject to $g(x,y,z) = 0$.
Because of the symmetry involved in working out the derivatives we need only do one in detail and populate the remaining two appropriately.

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{(\sqrt{x}+\sqrt{y}+\sqrt{z})^2 - (x+y+z)(\sqrt{x}+\sqrt{y}+\sqrt{z})\,\frac{2}{2\sqrt{x}}}{(\sqrt{x}+\sqrt{y}+\sqrt{z})^4} \\ &= \frac{\sqrt{x}+\sqrt{y}+\sqrt{z} - (x+y+z)\,\frac{1}{\sqrt{x}}}{(\sqrt{x}+\sqrt{y}+\sqrt{z})^3}\end{aligned} \tag{9}$$

Similarly:

$$\frac{\partial f}{\partial y} = \frac{\sqrt{x}+\sqrt{y}+\sqrt{z} - (x+y+z)\,\frac{1}{\sqrt{y}}}{(\sqrt{x}+\sqrt{y}+\sqrt{z})^3} \tag{10}$$

and

$$\frac{\partial f}{\partial z} = \frac{\sqrt{x} + \sqrt{y} + \sqrt{z} - (x + y + z)\frac{1}{\sqrt{z}}}{(\sqrt{x} + \sqrt{y} + \sqrt{z})^3} \tag{11}$$

From:

$$\nabla f = \lambda \nabla g \tag{12}$$

we have that:

$$\begin{aligned} \frac{\partial f}{\partial x} &= \lambda \\ \frac{\partial f}{\partial y} &= \lambda \\ \frac{\partial f}{\partial z} &= \lambda \end{aligned} \tag{13}$$

or $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = \frac{\partial f}{\partial z}$.

Now from $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y}$ we obtain $\frac{1}{\sqrt{x}} = \frac{1}{\sqrt{y}}$ or $x = y$. And similarly from $\frac{\partial f}{\partial y} = \frac{\partial f}{\partial z}$ we have that $y = z$. From the constraint $x + y + z = 1$ we must have that the minimum occurs at $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. The minimum is therefore:

$$f(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) = \frac{1}{\left(\frac{3}{\sqrt{3}}\right)^2} = \frac{1}{3} \tag{14}$$

There is a lot of symmetry going on here!

3 References

[1] https://johncarlosbaez.wordpress.com/2021/04/04/the-koide-formula/#comments

[2] https://www.researchgate.net/publication/2015951_The_strange_formula_of_Dr_Koide/link/53f1e6330cf272810e4c72b6/download

4 History

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