The basics of trigonometric integration

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1 The importance of trigonometric integration

It is no exaggeration to say that trigonometric integration underpins the operation of modern society. This may seem a bold claim but a little thought about applications of trigonometric integration will convince you that it is a valid claim. High school teachers who seem to know absolutely nothing about applications of what they teach would leave you with the impression that trigonometric integration is banal and useless.

The crown in the jewel of trigonometric integration is, of course, Fourier theory which is a vast subject that has penetrated every aspect of modern life. When you make a telephone call, look at a photo or video, Fourier theory is involved. When you get x-rays, CT scans and so on, Fourier theory or its descendants are involved. The whole edifice of electrical engineering relies to varying degrees on Fourier theory. Fourier theory is used in all sorts of ways in the defence, music and many other industries. Even something as abstract as number theory involves Fourier theory both at a theoretical and practical level - an example of the latter being the algorithms used by the GIMPS Mersenne Prime Search which rely upon Fast Fourier Transforms to perform massive integer calculations.

In physics, trigonometric integration is literally everywhere. Historically, Fourier's use of Fourier series in heat diffusion problems was generalised to many other contexts. If you want to know why light appears to travel in a straight line you need to understand the behaviour of the Feynman path integral which involves trigonometric integration (explained in more detail below).

Finally, trigonometric integration is actually fundamental to probability theory via the concept of the characteristic function (Fourier theory raises its head yet again!). The works of Levy and Bochner are full of Fourier theory. In what follows from section 3 onwards I will use the foundational approach employed by Salomon Bochner in his "Lectures on Fourier Integrals" [8]. It is an "old style" approach not followed these days and reminds me somewhat of the approach taken on functional analysis by Frigyes Riesz and Bela Sz.-Nagy [9]

More information on Fourier theory can be found in [1]

2 An example of the importance of trigonometric integration in physics - the Feynman path integral

In his book "QED: The StrangeTheory of Light and Matter" Richard Feynman gives a high level explanation of why, among other things, light appears to travel in straight lines ([3], Chapter 2]. In that discussion he refers to some paths adding up and some paths cancelling. In his more technical textbook ([2], page 77) he gives a more detailed explanation of what is going on. The starting point is a a path integral of the following form:

$$\psi(x,t+\epsilon) = \frac{1}{A}\int_{-\infty}^{\infty} e^{\frac{i}{\hbar}\epsilon\mathcal{L}\left(\frac{x-y}{\epsilon},\frac{x+y}{2}\right)}\,\psi(y,t)\,dy \tag{1}$$

Here $\mathcal{L}$ is the Lagrangian and we only need to focus on the form of the integrand when we apply (1) to the special case of a particle moving in one dimension subject to a potential energy $V(x,t)$. In such a case the Lagrangian, being the difference between kinetic and potential energy is:

$$\mathcal{L} = \frac{1}{2}m\dot{x}^2 - V(x,t) \tag{2}$$

Thus (1) becomes:

$$\psi(x,t+\epsilon) = \frac{1}{A}\int_{-\infty}^{\infty} e^{\frac{i}{\hbar}\frac{m(x-y)^2}{2\epsilon}} \times e^{-\frac{i}{\hbar}\epsilon V\left(\frac{x+y}{2},t\right)}\,\psi(y,t)\,dy \tag{3}$$

Now if you are trying to reconcile (2) with (3), you need to have read all the lead up to (2). Feynman is doing an estimate of an integral and in this context the velocity is:

$$\dot{x} = \frac{x_{i+1}-x_i}{\epsilon} \tag{4}$$

and acceleration (see [2], page 34) is:

$$\ddot{x} = \frac{x_{i+1} - 2x_i + x_{i-1}}{\epsilon^2} \tag{5}$$

So with $x_{i+1} = x$ and $x_i = y$ you get (3). Equations (4) and (5) are simply numerical approximations of the relevant derivatives. After all, Feynman was a simple man!

Feynman goes on to explain the behaviour of (3) as follows ([3], page 77):

"The quantity $\frac{(x-y)^2}{\epsilon}$ appears in the exponent of the first factor. It is clear that if y is appreciably different from x, this quantity is very large and the exponential consequently oscillates very rapidly as y varies. When this factor oscillates rapidly, the integral over y gives a very small value (because of the smooth behavior of the other factors). Only if y is near x (where the exponential changes more slowly) do we get important contributions. For this reason we make the substitution $y = x + \eta$ with the expectation that appreciable contributions to the integral will occur only for small η. "
In fact Feynman goes on a page later to show that $\psi(x,t)$ satisfies the Schrodinger wave equation $\frac{\partial \psi}{\partial t} = -\frac{i}{\hbar}\Big[-\frac{\hbar^2}{2m}\frac{\partial^2 \psi^2}{\partial x^2} + V(x,t)\,\psi \Big]$.

We shall soon see how this property Feynman refers to can be proved quite generally and reflects a fundamental property of trigonometric integrals.

3 The basic integrals

We start with these integrals:

$$\phi(\alpha) = \int_a^b f(x)\,\cos\alpha x\,dx \tag{6}$$

$$\psi(\alpha) = \int_a^b f(x)\,\sin\alpha x\,dx \tag{7}$$

$$J(\alpha) = \int_a^b f(x)\,e^{i\alpha x}\,dx \tag{8}$$

In what follows we will focus on $J(\alpha) = \phi(\alpha) + i\psi(\alpha)$. The most important cases involve an integration from $-\infty$ to ∞ and we will come to that.

Some basic properties of these integrals as as follows:

$$\begin{aligned}\phi(\alpha)+i\psi(\alpha)=&\int_a^b f(x)\,\cos\alpha x\,dx+i\int_a^b f(x)\,\sin\alpha x\,dx\\=&\int_a^b f(x)\,e^{i\alpha x}\,dx\\=&J(\alpha)\end{aligned}\tag{9}$$

$$\begin{aligned}J(\alpha)+J(-\alpha)=&\int_a^b f(x)\,e^{i\alpha x}\,dx+\int_a^b f(x)\,e^{-i\alpha x}\,dx\\=&\int_a^b f(x)\big(e^{i\alpha x}+e^{-i\alpha x}\big)\,dx\\=&2\int_a^b f(x)\,\cos\alpha x\,dx\\=&2\phi(\alpha)\end{aligned}\tag{10}$$

$$\begin{aligned}J(\alpha)-J(-\alpha)=&\int_a^b f(x)\,e^{i\alpha x}\,dx-\int_a^b f(x)\,e^{-i\alpha x}\,dx\\=&\int_a^b f(x)\big(e^{i\alpha x}-e^{-i\alpha x}\big)\,dx\\=&2i\int_a^b f(x)\,\sin\alpha x\,dx\\=&2i\psi(\alpha)\end{aligned}\tag{11}$$

$$\begin{aligned}\phi(\alpha)=&\phi(-\alpha)\text{ evenness of cos}\\\psi(-\alpha)=&-\psi(\alpha)\text{ oddness of sin}\\J(-\alpha)=&\overline{J_1(\alpha)}\text{ where }J_1(\alpha)=\int_a^b\overline{f(x)}\,e^{i\alpha x}\,dx\end{aligned}\tag{12}$$

4 The fundamental trigonometrical property

Assuming a finite interval of integration as a starting point, we can say that **generally** $J(\alpha)$ will become arbitrarily small for large values of α. This principle gets its own box for emphasis because it is so fundamental:

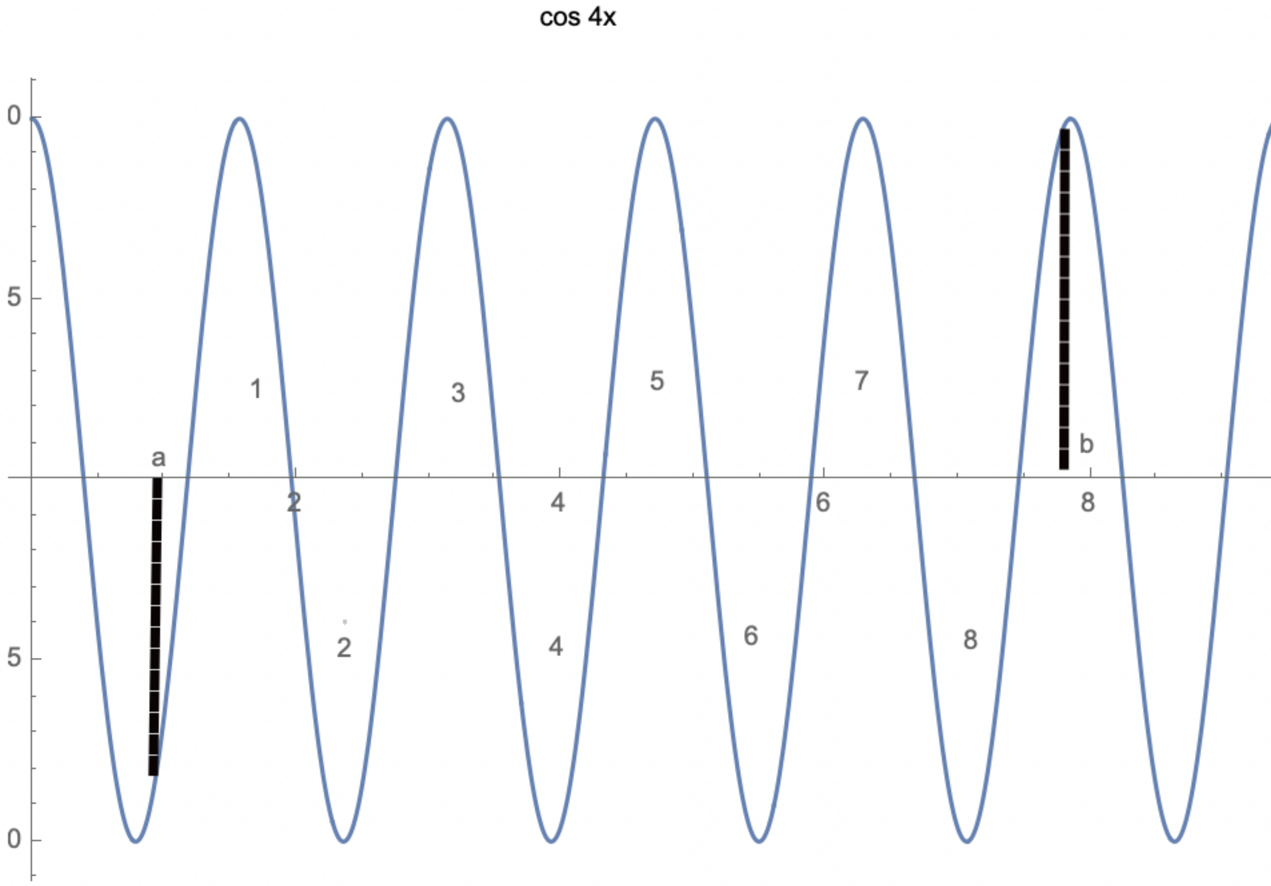
$$\boxed{J(\alpha) \to 0 \text{ as } \alpha \to \pm\infty} \tag{13}$$

It is reasonable to ask whether, at a high level, the fundamental property in (13) passes the "smell test". To approach this at a high level let us take a brutally simple case where $f(x)$ is a step function such that $f(x) = 1$ when $x \in (a, b)$ and 0 outside that interval. Thus $J(\alpha) = \int_a^b \cos \alpha x \, dx + i \int_a^b \sin \alpha x \, dx$.

Note: Now because $\sin \alpha = \cos(\frac{\pi}{2} - \alpha)$ we may as well just view $J(\alpha) = \int_a^b \cos \alpha x \, dx$ for the purposes of this high level analysis. Anything we say can be appropriately adjusted for the imaginary component. I will generally ignore it in what follows for this reason - the same arguments apply symmetrically to the imaginary component. The context will make it clear when I want to deal with both the real and imaginary components of the $J(\alpha)$.

In the figure below we can immediately see that $J(4) = \int_a^b \cos 4x \, dx$ is simply the sum of the two small slivers of area - one near a and one near b. You can see that in between there is a symmetrical offsetting of the signed areas ie the sum of odd numbered positive areas = the sum of even numbered negative areas. The net effect of the two "tail" areas is small. Indeed if we perform the general integration we get:

$$\begin{aligned}
\left| \int_a^b \cos \alpha x \, dx \right| &= \left| \frac{1}{\alpha} (\sin \alpha b - \sin \alpha a) \right| \\
&\leq \frac{1}{|\alpha|} (|\sin \alpha b - \sin \alpha a|) \\
&\leq \frac{1}{|\alpha|} (|\sin \alpha b| + |\sin \alpha a|) \\
&\leq \frac{1}{|\alpha|} \times 2 \\
&= \frac{2}{|\alpha|} \to 0 \text{ as } \alpha \to \pm\infty
\end{aligned} \tag{14}$$



You might say that is very well when $f(x)$ is a simple step-function, but what if $f(x)$ is more complicated? However, one can approximate an integrable function arbitrarily closely by appropriate step-functions. This involves a type of Stone-Weierstrass approximation. Littlewood proved this generally for Lebesgue integrable functions ([4], page 27). The logic applies to Riemann integrable functions in the sense that for bounded functions the Riemnann integral will equal the Lebesgue integral. So you can replicate this logic for a more complicated function.

Having seen that, at a high level, the fundamental principle is believable, we can now proceed to a proof and to do so we need to put some conditions on $f(x)$ so we can get the estimates to work. Thus we assume that $f(x)$ is differentiable on (a, b) and that both $f(x)$ and $\int_a^b |f'(x)|\, dx$ are bounded by B ie

$$
\begin{aligned}
|f(x)| <& B \\
\int_a^b |f^{'}(x)|\, dx <& B
\end{aligned}
\tag{15}
$$

With these assumptions we simply perform a standard integration by parts as follows:

$$
\begin{aligned}
|J(\alpha)| =& \Big| \int_a^b f(x) d\Big(\frac{1}{i\alpha} e^{i\alpha x}\Big) \Big| \\
=& \Big| \frac{1}{i\alpha}[f(b)e^{i\alpha b} - f(a)e^{i\alpha a}] - \frac{1}{i\alpha} \int_a^b f^{'}(x) e^{i\alpha x}\, dx \Big| \\
\leq& \Big| \frac{1}{i\alpha}[f(b)e^{i\alpha b} - f(a)e^{i\alpha a}] \Big| + \Big| \frac{1}{i\alpha} \int_a^b f^{'}(x) e^{i\alpha x}\, dx \Big| \\
\leq& \frac{1}{|\alpha|}(|f(b)| + |f(a)|) + \frac{1}{|\alpha|} \int_a^b |f^{'}(x)|\, dx \\
\leq& \frac{3B}{|\alpha|} \to 0 \text{ as } |\alpha| \to \infty
\end{aligned}
\tag{16}
$$

If the fundamental principle (13) is valid for (a, b) it also applies to any sub-interval eg $\int_a^b = \int_a^c + \int_c^b$, so that it is valid for piecewise differentiable functions and, in particular step-functions which are used to approximate a more complicated function. Differentiability on open-subintervals is important here so we don't get problems with the derivative at an end point.

5 How can we generalise the types of functions beyond step functions?

Let's suppose we have any integrable functions (you can think of this as Riemann integrable if you are not familiar with Lebesgue integration) $f(x)$ and $f_1(x)$ where $f_1(x)$ is close to $f(x)$ in this sense for for any $\epsilon > 0$:

$$
\int_a^b |f(x) - f_1(x)|\, dx < \epsilon \tag{17}
$$

Then with an obvious notation for the trigonometric integral of $f_1(x)$ we have:

$$\begin{aligned}|J(\alpha) - J_1(\alpha)| =& \left| \int_a^b \big(f(x) - f_1(x)\big) e^{i\alpha x}\, dx \right| \\ \leq& \int_a^b |f(x) - f_1(x)|\, dx < \epsilon\end{aligned} \tag{18}$$

If $f_1(x)$ satisfies our fundamental property (13) then we can find an α which depends on ϵ (so to emphasise this we write $\alpha(\epsilon)$) such that:

$$|J_1(\alpha)| < \epsilon \tag{19}$$

for $|\alpha| > \alpha(\epsilon)$. We can do this because, by assumption, $J_1(\alpha) \to 0$ as $|\alpha| \to \infty$. Hence for $|\alpha| > \alpha(\epsilon)$ we have, using a standard fiddle:

$$\begin{aligned}|J(\alpha)| =& |J_1(\alpha) + J(\alpha) - J_1(\alpha)| \\ \leq& |J_1(\alpha)| + |J(\alpha) - J_1(\alpha)| \\ <& \epsilon + \epsilon \\ =& 2\epsilon\end{aligned} \tag{20}$$

Thanks to Littlewood for each integrable function $f(x)$ and for any $\epsilon > 0$ we like we can conjure up a step-function $f_1(x)$ which is close to $f(x)$ in the sense of (17).

Thus for each integrable function $f(x)$ our fundamental proposition (13) holds. This logic clearly applies to $\phi(\alpha)$ and $\psi(\alpha)$ eg in (18) the estimate continues to hold because $|\sin \alpha x| <= 1$ and $|\cos \alpha x| <= 1$.

6 $J(\alpha)$ is continuous

To show that $J(\alpha)$ is continuous consider the difference:

$$\begin{aligned}\big|J(\alpha + \lambda) - J(\alpha)\big| =& \left| \int_a^b f(x)(e^{i(\alpha+\lambda)x} - e^{i\alpha x})\, dx \right| \\ \leq& \int_a^b |f(x)|\, |e^{i\alpha x}(e^{i\lambda x} - 1)|\, dx \\ =& \int_a^b |f(x)|\, |e^{i\lambda x} - 1|\, dx \\ \leq& B(\lambda) \int_a^b |f(x)|\, dx\end{aligned} \tag{21}$$

where $B(\lambda) = \max_{\lambda \in [a,b]} \left\{ e^{i\lambda x} - 1 \right\}$. Now, if $\lambda \to 0$ then $B(\lambda) \to 0$ and this means that the RHS of the last line of (21) can be made arbitrarily small - hence $J(\alpha)$ is continuous .

7 The integrals over infinite intervals

We just recall briefly what an infinite integral means. Take any $M > 0$, then the integral $\int_a^\infty g(x)\, dx$ has the meaning that:

$$\int_a^M g(x)\, dx \tag{22}$$

exists as a finite number for M no matter how big ie as $M \to \infty$.

Note that if $g(x)$ is integrable as $x \to \infty$ then it does not necessarily follow that $|g(x)|$ is also integrable as $x \to \infty$, although the converse is true ie the integrability of at $|g(x)|$ implies the integrability of $g(x)$. The classic example of this is the integral of the sinc function:

$$\int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2} \tag{23}$$

but

$$\int_0^\infty \left| \frac{\sin x}{x} \right| dx \text{ diverges} \tag{24}$$

Also (23) is Riemann integrable but **not** Lebesgue integrable, indeed it is the classic case of this phenomenon which might thought to be odd since Lebesgue integration arose from seeking to expand the types of functions that could be meaningfully integrated. This is not the place to go into the ins and outs of this. Suffice it to say that if $g(x)$ is Lebesgue integrable then so is $|g(x)|$, but if you can show that the integral of $|g(x)|$ does not converge, it cannot be Lebesgue integrable. In the case of $|\frac{\sin x}{x}|$ this is effectively done by comparison with the harmonic series. My paper on the Dirichlet kernel has estimate calculations in it which can be adjusted to demonstrate the point [5].

There are some straightforward observations we can make. One is that if $f(x)$ is absolutely integrable on $[a, \infty)$ ie $\int_a^\infty |f(x)|\, dx$ exists, then we will have for any α:

$$
\begin{aligned}
\int_a^\infty |f(x)\sin\alpha x|\,dx \leq& \int_a^\infty |f(x)|\,|\sin\alpha x|\,dx \\
\leq& \int_a^\infty |f(x)|\,dx \\
<&\infty
\end{aligned}
\tag{25}
$$

Thus $\psi(\alpha) = \int_a^\infty f(x)\sin\alpha x\,dx$ converges for all α. The same logic applies to $\phi(\alpha)$.

The next observation we can make is that $\psi(\alpha) \to 0$ as $|\alpha| \to \infty$. This follows from the usual trick of breaking up the domain of integration and showing that each bit is suitably small as follows:

$$
\begin{aligned}
|\psi(\alpha)| =& \left| \int_a^M f(x)\sin\alpha x\,dx + \int_M^\infty f(x)\sin\alpha x\,dx \right| \\
\leq& \left| \int_a^M f(x)\sin\alpha x\,dx \right| + \left| \int_M^\infty f(x)\sin\alpha x\,dx \right| \\
\leq& \left| \int_a^M f(x)\sin\alpha x\,dx \right| + \underbrace{\int_M^\infty |f(x)|\,dx}_{\text{independent of } \alpha \text{ and can be made small by making M large}} \\
<& \epsilon + \epsilon = 2\epsilon
\end{aligned}
\tag{26}
$$

Because of the absolute integrability of $f(x)$ the second integral in the last line of (26) can be made less than ϵ by simply going far enough out ie making M big. Now with M fixed (it has been chosen suitably big so that the second integral is less than ϵ) and $|\alpha| > \alpha(\epsilon)$ (to emphasise that α depends on the choice of ϵ) we can also make the first integral less than ϵ bearing in mind the discussion about how the fundamental property in (13) applies to any integrable function. For reasons already given we can apply this logic to $\phi(\alpha)$ and hence $J(\alpha)$.

One technical point to note here is that you don't want to push the absolute value signs through the first integral since that results in the sum of a lot of positive amounts and that may not be small!

7.1 A simple example

Let $f(x) = e^{-kx}$ for $k > 0, a = 0$.

$$
\begin{aligned}
J(\alpha) &= \int_0^M e^{-(k-i\alpha)x}\,dx \\
&= -\frac{1}{k-i\alpha}\left[e^{-(k-i\alpha)x}\right]_0^M \\
&= \frac{1}{k-i\alpha}\left[1 - e^{-(k-i\alpha)M}\right] \\
&= \frac{k+i\alpha}{k^2+\alpha^2}\left[1 - e^{-kM}(\cos\alpha M + i\sin\alpha M)\right] \\
&= \frac{k+i\alpha}{k^2+\alpha^2}\left[1 - e^{-kM}\cos\alpha M - ie^{-kM}\sin\alpha M)\right] \\
&= \frac{1}{k^2+\alpha^2}\Big[k(1 - e^{-kM}\cos\alpha M - ie^{-kM}\sin\alpha M) + i\alpha(1 - e^{-kM}\cos\alpha M) \\
&\quad + \alpha e^{-kM}\sin\alpha M\Big] \\
&= \frac{1}{k^2+\alpha^2}\Big[\underbrace{k - ke^{-kM}\cos\alpha M + \alpha e^{-kM}\sin\alpha M}_{\text{real part}} + i\underbrace{(\alpha - \alpha e^{-kM}\cos\alpha M - ke^{-kM}\sin\alpha M)}_{\text{imaginary part}}\Big]
\end{aligned}
\tag{27}
$$

As $M \to \infty$, the real part approaches $\frac{k}{k^2+\alpha^2}$ and the imaginary part approaches $\frac{\alpha}{k^2+\alpha^2}$. Therefore we have:

$$\int_0^\infty e^{-kx}\cos\alpha x\,dx = \frac{k}{k^2+\alpha^2} \tag{28}$$

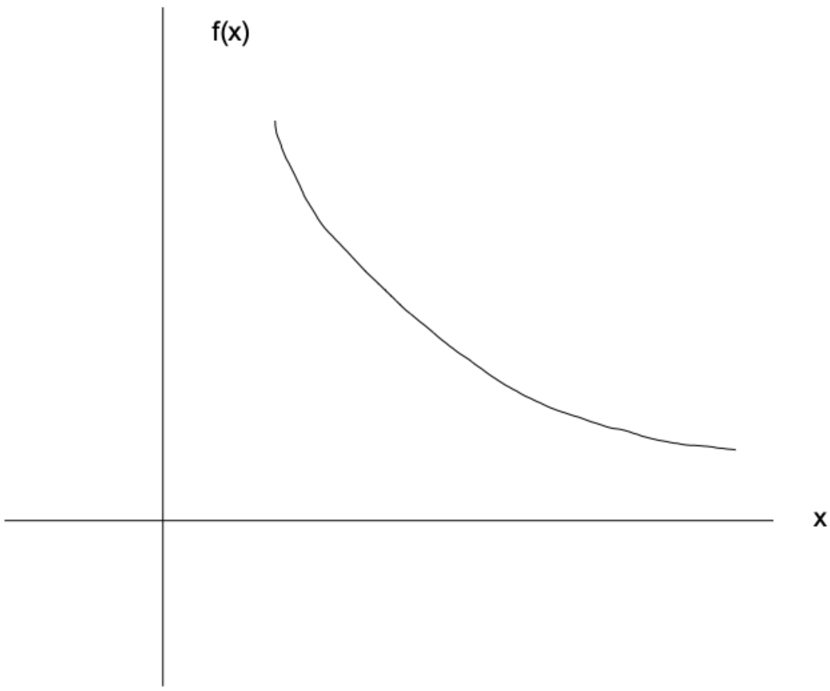
$$\int_0^\infty e^{-kx}\sin\alpha x\,dx = \frac{\alpha}{k^2+\alpha^2} \tag{29}$$

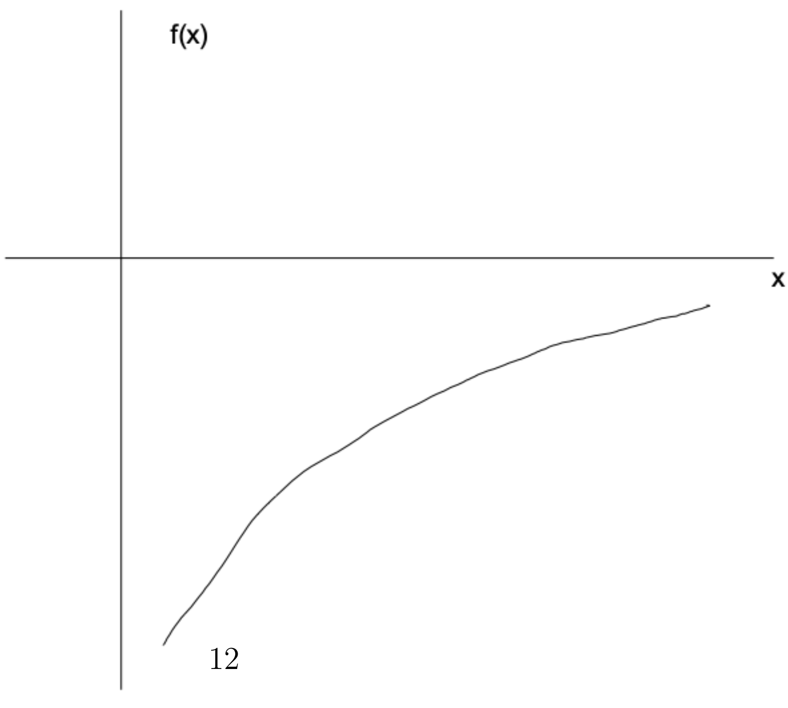
You can also obtain (28) and (29) by double integration by parts in each case.

Note also that from (28) and (29), for fixed k, as $|\alpha| \to \infty$, $\int_0^\infty e^{-kx}\cos\alpha x\,dx \to 0$ and $\int_0^\infty e^{-kx}\sin\alpha x\,dx \to 0$ as our theory advertised.

8 The special case of monotonic functions

Absolute integrability is not necessary if the function is monotonic. If $f(x)$ converges monotonically to zero as $x \to \infty$, you can have either of these two scenarios:


f(x)
x


f(x)
x

The bottom case is just the negative of the top case, so we can just focus on the top case. We can get an estimate for the integral of the product of a positive monotonically decreasing function and a continuous function by using the so-called "second mean value of integral calculus". This second mean value theorem is also known as the Bonnet's mean value theorem and is proved in [6], pages 244-46. The form of Bonnet's mean value theorem we need is this:

If on the interval (a, b) the function $g(x)$ is continuous and $f(x)$ is positive and monotonically decreasing, then in (a, b) there is a value c between a and b such that:

$$\int_a^b f(x)g(x)\,dx = f(a)\int_a^c g(x)\,dx \tag{30}$$

If we let $g(x) = \sin \alpha x$ for $\alpha > 0$ we can see by the following estimates that:

$$\psi(\alpha) = \int_a^\infty f(x)\sin \alpha x\,dx \tag{31}$$

converges for $\alpha > 0$.

We have that:

$$\begin{aligned} \Big|\int_a^c \sin \alpha x\,dx\Big| &= \frac{1}{\alpha}|-\cos \alpha c + \cos \alpha a| \\ &\leq \frac{1}{\alpha}\big(|\cos \alpha c| + |\cos \alpha a|\big) \\ &\leq \frac{2}{\alpha} \end{aligned} \tag{32}$$

Now using the Bonnet second mean value theorem (30) we see that for $c \in (a, b)$:

$$\begin{aligned} \Big|\int_a^b f(x)\sin \alpha x\,dx\Big| &= \Big|f(a)\int_a^c \sin \alpha x\,dx\Big| \\ &\leq |f(a)|\,\Big|\int_a^c \sin \alpha x\,dx\Big| \\ &\leq \frac{2|f(a)|}{\alpha} \end{aligned} \tag{33}$$

If now $f(x)$ is positive and monotonically decreasing to zero in (a, ∞) then we will have for $a \leq M < N < \infty$:

$$\Big| \int_M^N f(x) \sin \alpha x \, dx \Big| \le \frac{2|f(M)|}{\alpha} \tag{34}$$

and since $f(M) \to 0$ as $M \to \infty$ it follows that:

$$\psi(\alpha) = \int_a^\infty f(x) \sin \alpha x \, dx \tag{35}$$

is convergent for $\alpha > 0$ and, as before, $\psi(\alpha) \to 0$ as $\alpha \to \pm\infty$. Thus we have the following theorem:

If in $[a, \infty)$ the function $f(x)$ has the following properties as $x \to \infty$, either:
1. it is absolutely integrable, or
2. converges monotonically to zero,
then the integrals $\phi(\alpha), \psi(\alpha), J(\alpha)$ exist for:
3. all α or
4. all $\alpha \neq 0$,
and converge to zero as $\alpha \to \pm\infty$.

It is to be noted that the restriction $\alpha \neq 0$ referred to in 4. above only applies to $\phi(\alpha)$ and $J(\alpha)$ and not to $\psi(\alpha)$. If the function $f(x)$ is monotonically decreasing eg $f(x) = \frac{1}{x}$ it is not necessary that the integral $\int_a^\infty f(x)\, dx$ represent the value $\phi(0)$ or $J(0)$, since in that case the integral would not converge, but the analysis set out above and summarised in the theorem shows what happens as $\alpha \to \pm\infty$.

8.1 The behaviour of $\int_a^\infty \frac{\sin \alpha x}{x} \, dx$

This integral was pivotal to Dirichlet's work in the 19^{th} century on Fourier series.

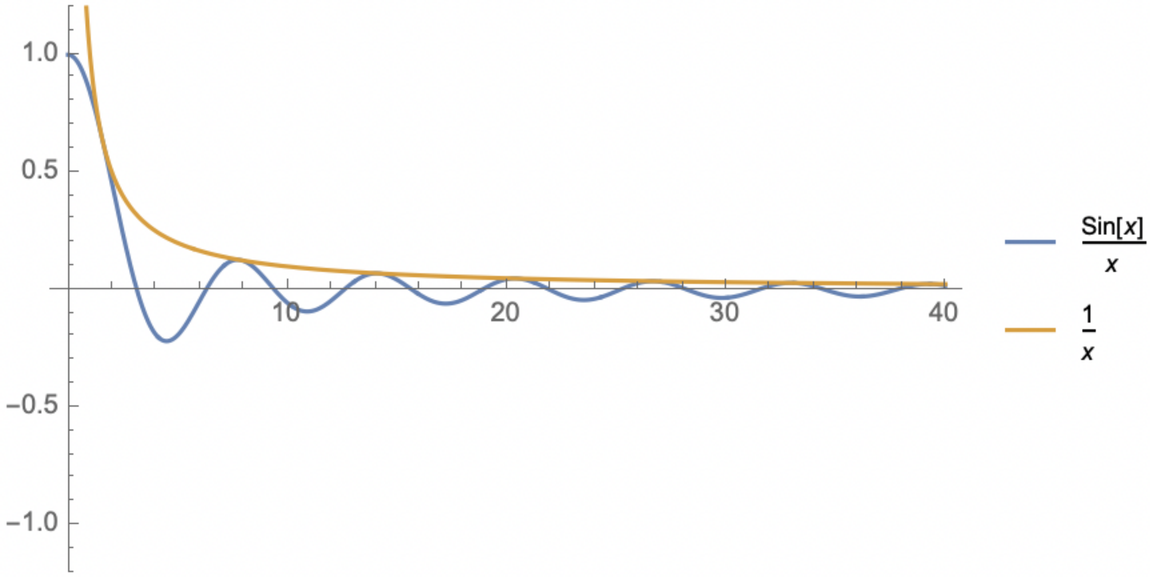
For $a > 0$,

$$\int_a^\infty \frac{\sin \alpha x}{x} \, dx \tag{36}$$

is covered by the theorem because the function $f(x) = \frac{1}{x}$ decreases monotonically to zero in the interval (a, ∞) but is not integrable on $[0, a]$ or indeed $[0, \infty)$. So we can say that

$$\psi(\alpha) = \int_a^\infty \frac{\sin \alpha x}{x} \, dx \tag{37}$$

exists for all α but unfortunately it does not converge to zero as $\alpha \to \infty$. The following graph exemplifies the relevant behaviour.



Now for $\alpha > 0$ of we make the substitution $\alpha x = \xi$ and we get:

$$\int_0^\infty \frac{\sin \alpha x}{x}\, dx = \int_0^\infty \frac{\sin \xi}{\xi}\, d\xi \tag{38}$$

and this means that $\psi(\alpha)$ is constant for $\alpha > 0$ and, as is known from elementary calculus, this constant is not zero - it is in fact $\frac{\pi}{2}$. A proof which deals with the alternating signs of the relevant areas is set out in the Appendix in [1].

The final topic of this basic overview is the concept of uniform convergence of trigonometric integrals.

9 Uniform convergence of trigonometric integrals

We start with a convergent integral:

$$\int_a^\infty g(x, \lambda)\, dx \tag{39}$$

which depends on the parameter λ. This integral is uniformly convergent if for each $\epsilon > 0$ one can find an $A(\epsilon)$ (to emphasise the dependence of A on ϵ) such that for all $A > A(\epsilon)$, and for all relevant values of λ we have:

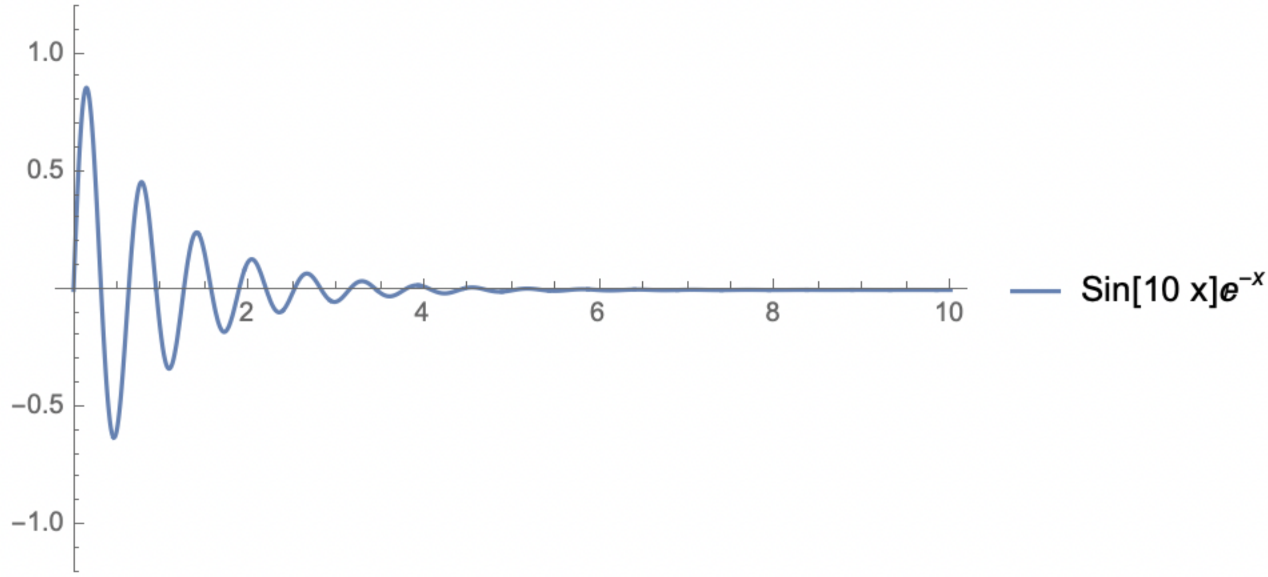
$$\Big| \int_A^\infty g(x,\lambda)\, dx \Big| < \epsilon \tag{40}$$

Clearly, for the purposes of this definition, the lower limit can be $-\infty$. If we can find an absolutely integrable function $\gamma(x)$ such that $|g(x,\lambda)| < |\gamma(x)|$ we are home free.

In the case of $\psi(\alpha)$ we will have the following for an absolutely integrable function $f(x)$:

$$\begin{aligned} |\psi(\alpha)| =& \Big| \int_A^\infty f(x) \sin \alpha x \, dx \Big| \\ \leq& \int_A^\infty |f(x) \sin \alpha x| \, dx \\ \leq& \int_A^\infty |f(x)| \, dx \\ <& \epsilon \end{aligned} \tag{41}$$

since $f(x)$ is absolutely integrable. The estimates in (41) apply whether $A = 0, -\infty$. Note that in this case the behaviour of the parameter α is not relevant because of the absolute integrability of $f(x)$. Thus if $f(x) = e^{-x}$, which is absolutely integrable, quickly washes away the sinusoidal character of $\sin 10x$ as can be seen from the plot below:



The series of estimates in (31)-(35) show that if we choose $|\alpha|$ sufficiently large ie such that $|\alpha| > \alpha_0 > 0$ we satisfy the definition of uniform convergence. As before the reasoning can be applied symmetrically to $\phi(\alpha)$. The following properties of uniformly convergent integrals are what enable some very useful manipulations and are proved in most advanced calculus courses or analysis courses:

Property 1. If $g(x, \lambda)$ is continuous for $a \leq x < \infty$ and $\lambda_0 \leq \lambda \leq \lambda_1$ and if the integral:

$$G(\lambda) = \int_a^\infty g(x, \lambda)\, dx \tag{42}$$

converges uniformly, then $G(\lambda)$ is continuous in (λ_0, λ_1).

Property 2.

$$\int_{\lambda_0}^{\lambda_1} G(\lambda)\, d\lambda = \int_a^\infty \Big\{ \int_{\lambda_0}^{\lambda_1} g(x, \lambda)\, d\lambda \Big\}\, dx \tag{43}$$

Property 3. If $g(x, \lambda)$ is differentiable at every point with respect to λ and if the function

$$\frac{\partial g(x, \lambda)}{\partial \lambda} \tag{44}$$

is itself continuous and possesses a uniformly convergent integral, then $G(\lambda)$ is differentiable and

$$G^{'}(\lambda) = \int_a^\infty \frac{\partial g(x, \lambda)}{\partial \lambda}\, dx \tag{45}$$

The following example shows how these properties can be used to prove that $\int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}$.

9.1 Proving that $\int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}$

Let us start by integrating (28) ie$\int_0^\infty e^{-kx} \cos \alpha x\, dx = \frac{k}{k^2+\alpha^2}$, with respect to α between the limits 0 and α. Thus we have for the left hand side, using Fubini's Theorem - see [7] pages 75-86:

$$
\begin{aligned}
\int_0^\alpha \int_0^\infty e^{-kx} \cos \alpha x \, dx \, d\alpha &= \int_0^\infty \left(\int_0^\alpha e^{-kx} \cos \alpha x \, d\alpha \right) dx \\
&= \int_0^\infty \left(\int_0^\alpha e^{-kx} \, d\left(\frac{\sin \alpha x}{x} \right) \right) dx \\
&= \int_0^\infty \left[e^{-kx} \frac{\sin \alpha x}{x} \bigg|_{\alpha=0}^{\alpha} - \int_0^\alpha \frac{\sin \alpha x}{x} \underbrace{\frac{de^{-kx}}{d\alpha}}_{=0} \right] dx \\
&= \int_0^\infty e^{-kx} \frac{\sin \alpha x}{x} \, dx
\end{aligned}
\tag{46}
$$

Integrating the right hand side of (28) with respect to α gives, for $k > 0$ and the substitution $u = \frac{\alpha}{k}$:

$$
\begin{aligned}
\int_0^\alpha \frac{k}{k^2 + \alpha^2} \, d\alpha =& \frac{1}{k} \int_0^\alpha \frac{1}{1 + \left(\frac{\alpha}{k}\right)^2} \, d\alpha \\
=& \frac{1}{k} \int_0^{\frac{\alpha}{k}} \frac{k}{1 + u^2} \, du \\
=& \arctan\left(\frac{\alpha}{k} \right)
\end{aligned}
\tag{47}
$$

Hence we have:

$$
\int_0^\infty e^{-kx} \frac{\sin \alpha x}{x} \, dx = \arctan\left(\frac{\alpha}{k} \right) \tag{48}
$$

Fortunately Mathematica agrees with all of this!

In[∘]:= $\int_0^{\alpha}\left(\int_0^{\infty} e^{-k x}\,\mathrm{Cos}[\alpha\, x]\,dx\right)d\alpha$

Integrate: Unable to prove that integration limits $\{0, \alpha\}$ are real. Adding assumptions m

Out[∘]= $\int_0^{\alpha} \boxed{\frac{k}{k^2+\alpha^2} \quad \text{if} \quad \mathrm{Im}[\alpha] < \mathrm{Re}[k]\ \&\&\ \mathrm{Re}[k] > 0}\, d\alpha$

In[∘]:= $\int_0^{\infty}\left(\int_0^{\alpha} e^{-k x}\,\mathrm{Cos}[\alpha\, x]\,d\alpha\right)dx$

Out[∘]= $\boxed{\mathrm{ArcTan}\left[\frac{\alpha}{k}\right] \quad \text{if} \quad \mathrm{Re}[k] > 0}$

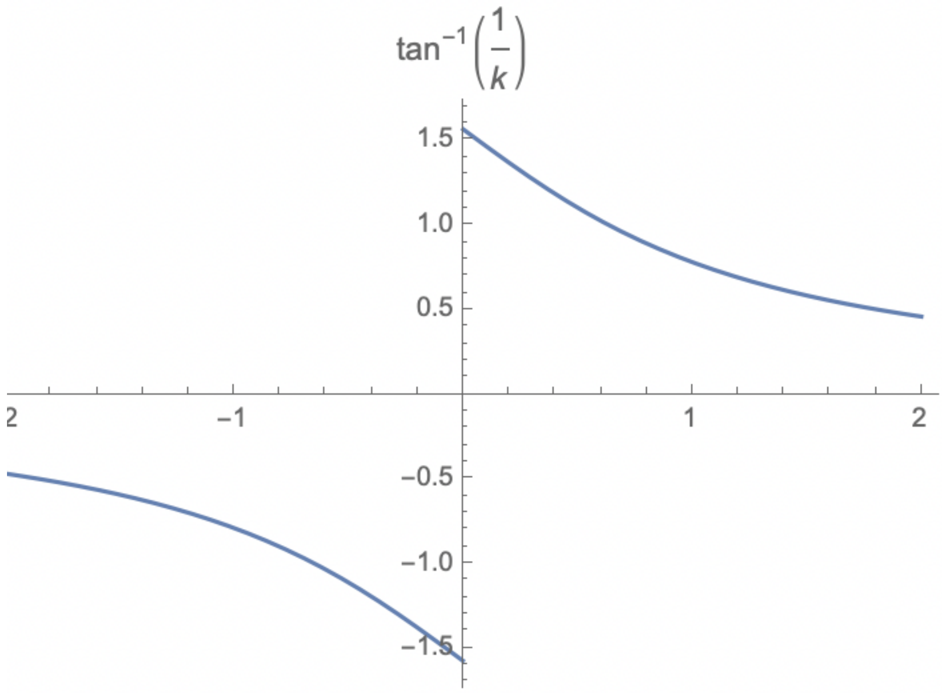
To prove that $\int_0^\infty e^{-kx}\frac{\sin\alpha x}{x}\,dx$ is uniformly convergent consider any $A > 0$, $M > 0$ and fixed $\alpha > 0$. Now for any $0 \le k < \infty$ we have, noting that for $A > 0$, $\frac{e^{-kx}}{x}$ monotonically decreases to zero as $A \to \infty$ (just differentiate and look at the result to convince yourself):

$$
\begin{aligned}
\left|\int_A^M \frac{e^{-kx}}{x}\sin\alpha x\,dx\right| &\le \frac{e^{-kA}}{A}\left|\int_A^M \sin\alpha x\,dx\right| \\
&= \frac{e^{-kA}}{A}\left|\frac{1}{\alpha}(-\cos M\alpha + \cos A\alpha)\right| \\
&\le \frac{e^{-kA}}{A}\,\frac{1}{\alpha}(|-\cos M\alpha| + |\cos A\alpha)|) \\
&\le \frac{e^{-kA}}{A}\,\frac{2}{\alpha} \\
&\le \frac{1}{A}\,\frac{2}{\alpha}
\end{aligned}
\tag{49}
$$

Thus if M is arbitrarily large the RHS of the last line of (49) shows that we can make the LHS as small as we want by making A large enough independent of k. Thus $\int_0^\infty e^{-kx}\frac{\sin\alpha x}{x}\,dx$ is uniformly convergent. Having established uniform convergence we can use Property 1 (42) as follows (using (48) with $\alpha = 1$):

$$
\begin{aligned}
\int_0^\infty \frac{\sin x}{x}\,dx &= \lim_{k\to 0}\int_0^\infty e^{-kx}\frac{\sin x}{x}\,dx \\
&= \lim_{k\to 0}\arctan\left(\frac{1}{k}\right) \\
&= \frac{\pi}{2}
\end{aligned} \tag{50}
$$

Noting here that $k \geq 0$ so the limit is $+\frac{\pi}{2}$ rather than $-\frac{\pi}{2}$ - see the graph below:



10 Final comments

This has been a introduction to some of the basic elements of trigonometrical integrals. There is much, much more! Fourier analysis is of course littered with trigonometric integrals so the ability to manipulate them is crucial to doing theoretical and practical work in that area. If you are keen you can plumb the depths of Zygmund's tome [9]. Zygmund was the late, great Eli Stein's doctoral supervisor as well as supervisor for Guido and Mary Weiss. He was also responsible for getting Alberto Calderon to Chicago University. This is only meaningful for harmonic analysis snobs but I'll throw it in anyway.

11 References

[1] Peter Haggstrom, Basic Fourier Integrals ,https://gotohaggstrom.com/Basic%20Fourier%20integrals.pdf

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[6] David Bressoud, A Radical Approach to Real Analysis ,Second Edition, Mathematical Association of America, 2007.

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[9] Frigyes Riesz and Bela Sz.-Nagy, Functional Analysis, Dover, 1990.

[10] A Zygmund, Trigonometric Series , Third Edition, Volumes I and II combined, Cambridge University Press, 2002.

12 History

Created 23 May 2020
27 May 2020 - corrected a transposition of words relating to the relationship between Riemann integrals and Lebesgue integrals. I should be shot.