peter.haggstrom@exemail.com.au
www.gotohaggstrom.com

The fallacy of time diversification

- a concept that financial planners do not really understand

What is the fallacy of time diversification?

■ The argument

Financial planners, journalists and investment pundits regularly state that as time increases the standard deviation of the annualised return decreases. While this is true (as will be demonstrated below) the use to which it is put is often fundamentally misleading. For instance, the Vanguard group in the US said in one of its publications that"the volatility of stock market returns diminishes markedly over time...Clearly, over time, stock market risk hardly seems excessive even for the most cautious investor" (Spring 1990 issue of Vanguard's US publication "*In The Vanguard*"). Financial planners and others argue from the decrease in the standard deviation of the annualised return (an average), that the uncertainty (which they equate either explicitly or implicitly with the standard deviation of the annualised return) of investing in volatile assets like stocks reduces as the time horizon increases.

However, investors are not interested in the standard deviation of the annualised return. Rather they are interested in their end investment value and it is the uncertainty associated with this that really matters. In essence those who peddle this argument are really asserting that the appropriate measure of the variability of the end investment is the standard deviation of the annualised return. Unwittingly some proponents of the argument may be signing up to the following claim:

standard deviation of the annualised return = standard deviation of the total return.

For a time scale of more than 1 year this equation is false if you take the average return as the annualised return. The theoretical underpinnings for the most general case are quite detailed and have been examined by a number of substantial writers such as Paul Samuelson, Robert Merton, Harry Markowitz and Stanley Fischer, just to mention the well known people. Significantly the seminal article by Merton and Samuelson carries the title: "*Fallacy of the Log-Normal Approximation to Portfolio Decision-Making Over Many Periods*". They used expected utility maximisation principles in their argument. As you might surmise, it is a non-trivial exercise to understand the detailed argument and that is why the superficial sales "argument" is so attractive to many people - it relieves you of the hard work of understanding the nuts and bolts.

Investors need to understand the basic point that time does not wash away all sins. I believe it is precisely because a lot of sales people have conditioned investors to believing that the uncertainty surrounding their end benefit will decay away to nothingness over time that investors are highly disturbed when they actually see large fluctuations in their portfolio's value.

There are still many good reasons for investing some of your money in equities and a lot of academic research has been done on the subject. However, what is clear from the theoretical standpoint is that you can't naively believe that volatility in your end investment value will disappear over time.

Understanding the mathematics

We start with a simplified situation to demonstrate the threshold error in the logic described above.

Suppose the continuously compounded annual rate of return on an investment has standard deviation σ. The investment horizon is t years and r_i is the rate of return during year i. For computational convenience it is assumed that the r_i are independent with the same standard deviation σ. This makes the maths nice and clear so the general point is not obscured. The average return is: $\frac{1}{t}\sum_{i=1}^{t} r_i$. The total return is $\sum_{i=1}^{t} r_i$. The development of the general argument is considerably more detailed than this and you can see the Appendix for the main papers on the subject. The Merton-Samuelson paper is over 100 pages.

Now compare the standard deviations of these two returns.

The standard deviation of the average return is:

$$\sqrt{\mathrm{Var}\left(\frac{1}{t}\sum_{i=1}^{t} r_i\right)} = \sqrt{\frac{1}{t^2}\mathrm{Var}\left(\sum_{i=1}^{t} r_i\right)}$$

$$= \sqrt{\frac{1}{t^2}\, t\, \sigma^2}$$

$$= \frac{\sigma}{\sqrt{t}}$$

Here "Var" means the variance and the standard deviation is the square root of the variance. The manipulations above rely upon the properties of the variance operator and can be found in any textbook.

Clearly as t increases the standard deviation of the average return decreases and approaches zero in the limit. Thus the standard deviation of the annualised return does decrease with increasing time.

The standard deviation of the total return is:

$$\sqrt{\mathrm{Var}\left(\sum_{i=1}^{t} r_i\right)} = \sqrt{t\,\sigma^2}$$

$$= \sqrt{t}\,\sigma$$

In this case the standard deviation of the total return **increases** with time. For a time scale of more than 1 year the purported equation is false.

In fact the standard Brownian motion models that underpin the entire theoretical edifice involve the assumption that returns follow a geometric Brownian motion with standard deviation $\sigma\sqrt{t}$ and mean (μ - $\frac{\sigma^2}{2}$) t. Thus the underlying models involve **increased** return variation with time so that those who argument that time reduces uncertainty have to get over that analytical fact.

While portfolio theory demonstrates that diversification of various sorts of assets that are not perfectly correlated can result in a decreased standard deviation of the expected annual portfolio return, the theoretical basis for that result revolves around **weightings** of assets. Those who peddle the time diversification fallacy are essentially translating the logic of weighting of assets to weighting the time dimension of the problem. As demonstrated above the problem does not "scale" in this fashion and the reasoning is fundamentally flawed.

This is a classic case of sales people pushing a product (invest in equities and it will be alright in the long run) without having any understanding whatsoever of what is really going on. The deeper theoretical understanding takes much more mathematical overhead to develop which naturally excludes the vast majority of the advisory industry. I will come to an overview of that deeper understanding in an appendix.

Making it concrete

To make the principles concrete we can take a brutally simple case where the rate of return in any one year period is either +20% or -20%. The worst possible outcome, indeed the catastrophic outcome, is where for t years the investment suffers a loss of 20% each year. What happens as the time scale increases?

Let V(0) be the value of the investments at time t = 0. After one year the value of the investment is V(1) = V(0) - 0.20 V(0) = 0.8 V(0). After 2 years the value of the investment is V(2) = V(1) =- 0.2 V(1) = 0.8 V(1) = 0.8^2 V(0).

One can proceed inductively to see that after time t the value of the investment is V(t) = 0.8^tV(0) so that after 20 years, say, the value of the investment is : V(20) = 0.8^{20} V(0) = 0.01 V(0). So after 20 years you have lost 99% of your initial investment! By assigning probabilities to the returns one can attach probabilties to such outcomes.

A widely used concept of risk is that of a shortfall which occurs when the value of the share portfolio at the horizon date falls below some value determined by a specified target rate of return. Such a target rate of return is usually a risk free rate. If it actually turns out that the mean rate of return exceeds the risk-free rate of interest, it is true that the probability of a shortfall declines with the length of the investment time horizon.

The problem is that the probability of a shortfall is a flawed measure because it completely ignores how **large** the potential shortfall may be. As indicated in the example above, using a probability of shortfall as the measure of risk, no distinction is made between a 20% loss and a 99% loss, since both are, by definition, shortfalls.

APPENDIX

In portfolio theory the variance of the portfolio's expected return is:

$\mathrm{Var}_p = \sum_{i=1}^{t} \sum_{j=1}^{t} w_i\, w_j\, \mathrm{Cov_{ij}}$

where w_i and w_j are the portfolio weights of assets i and j and $\mathrm{Cov_{ij}}$ is the covariance between the rates of return on assets i and j.

In a two asset world with weights w and 1-w the standard deviation of the portfolio expected return is:

$$\begin{aligned} \sigma_p &= \sqrt{w^2\, \sigma_1{}^2 + (1-w)^2\, \sigma_2{}^2 + 2\, w(1-w)\, \sigma_1\, \sigma_2\, \mathrm{Cov}_{12}} \\ &\leq w\sigma_1 + (1\text{-}w)\sigma_2 \end{aligned}$$

Thus in an equally weighted portfolio (ie 50%/50%) of two assets with σ_1 = 5% and σ_2 = 10% the portfolio standard deviation of expected return is less than 50% x 5% + 50% x 10% = 7.5%. Thus diversification has reduced the standard deviation of the portfolio's expected annual return but this is not the same as demonstrating that a longer time horizon has reduced the standard deviation of the portfolio's expected total return.

The general problem has been examined extensively:

1. R C Merton and P A Samuelson, "Fallacy of the Log-Normal Approximation to Portfolio Decision-Making Over Many Periods", *Journal of Financial Economics* (March 1974), 67-94
2. P A Samuelson, "Risk and Uncertainty: A Fallacy of Large Numbers", *Scientia (*April - May 1963), 1-6
3. P A Samuelson, "The Fallacy of Maximizing the Geometric Mean in Long Sequences of Investing or Gambling", *Proceedings of the National Academy of Sciences* (1971), 207 -21.

There are many more recent papers dealing with the issue.

As noted above there is quite a lot of mathematical overhead in understanding the detailed results but it is possible to sketch the line of argument.

One starts with the concept of a shortfall which is the amount by which the value of an equity portfolio is less than one based on some target rate of return such as the interest rate on a default free zero coupon bond. The risk free rate provides a benchmark against which to assess risky equities. This seems to be a reasonable basis upon which to model the relevant behaviour.

If it were true that equities were less risky in the long run, then the cost of insuring against earning less than the risk-free rate of interest ought to decline as the length of the investment horizon increases. But this is not the case - the opposite is true.

The argument runs like this.

Suppose the risk free rate is r and the current value of the portfolio is S. The continuously compounded yearly return is assumed to have standard deviation σ. Let the future value of S after t years at rate r be V. Then V = S e^{rt}.

Next we can do one of two things: let the money stay in the portfolio or put it all in the bank. If we keep the money in the portfolio the risk we are really measuring is that of not doing as well as putting the money in the bank.

What would it cost to insure against ending up with less than V at time t? Such an insurance policy would cover the difference between the terminal value of the bank investment and the equity investment. A European put option on the portfolio with a strike price V and time to expiration t can provide the required insurance and the cost of this option is the price of the insurance we want.

If the value of the required put option is P and the corresponding call option is C (having the same strike price and expiration date), the Put-Call Parity Theorem tells us that:

$P = C + V e^{-rt} - S = C + (S e^{rt}) e^{-rt} - S = C + S - S = C$

So the put and the call have the same price and you can use the Black-Scholes formula for the value of the call to get the value of the put.

Using the standard symbols eg N(.) is the normal distribution cumulative density function we have:

$$\begin{aligned}
P &= C \\
&= S\, N\left(\frac{\log(S/V) + \left(r + \frac{1}{2}\sigma^2\right)t}{\sigma\sqrt{t}}\right) - V e^{-rt} N\left(\frac{\log(S/V) + \left(r - \frac{1}{2}\sigma^2\right)t}{\sigma\sqrt{t}}\right) \\
&= S\, N\left(\frac{\log(S/V) + \left(r + \frac{1}{2}\sigma^2\right)t}{\sigma\sqrt{t}}\right) - S e^{rt} e^{-rt} N\left(\frac{\log(S/V) + \left(r - \frac{1}{2}\sigma^2\right)t}{\sigma\sqrt{t}}\right) \\
&= S\left[N\left(\tfrac{1}{2}\sigma\sqrt{t}\right) - N\left(-\tfrac{1}{2}\sigma\sqrt{t}\right)\right] \text{ by symmetry considerations}
\end{aligned}$$

This latter expression gives the area under the normal distribution curve between $-\frac{1}{2}\sigma\sqrt{t}$ and $+\frac{1}{2}\sigma\sqrt{t}$ and so obviously increases with increasing t. Hence the cost of insurance which is our proxy for the risk of the investment increases with the time horizon.

Peter Haggstrom
July 2010