# The magic of Carlson's inequality

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## 1 Abstract

This paper deals with Carlson's magical inequality in its discrete version. The discrete version of Carlson's inequality is as follows: If $a_1, a_2, \ldots$ are real numbers not all of which are zero, then $\left(\sum_{k=1}^{\infty} a_k\right)^4 < \pi^2\left(\sum_{k=1}^{\infty} a_k^2\right)\left(\sum_{k=1}^{\infty} k^2 a_k^2\right)$. The constant $\pi^2$ is sharp in the sense that if it is replaced by a smaller constant then there is a sequence of numbers $a_k$ for which the inequality is false. The continuous version of Carlson's inequality is: If $f$ is piecewise continuous on $\mathbb{R}_+$ then $\left(\int_0^{\infty} f(t)\,dt\right)^4 \leq \pi^2\left(\int_0^{\infty} f(t)^2\,dt\right)\left(\int_0^{\infty} t^2 f(t)^2\,dt\right)$. Note that in the continuous case there is not strict inequality.

Carlson's proof is long and difficult. He was an expert in analytic function theory and his proof reflects that background. Shortly after Carlson's result Hardy came along with a breathtakingly elementary proof that avoided all the complexity of Carlson's original proof.

## 2 Introduction

There are inequalities and then there are inequalities that take your breath away. You are sometimes left wondering how on earth someone came up with the idea let alone the proof. Here is a list of some of the most famous inequalities:

$$\sum_{k=1}^{n} a_k b_k \leq \left(\sum_{k=1}^{n} a_k^2\right)^{\frac{1}{2}}\left(\sum_{k=1}^{n} b_k^2\right)^{\frac{1}{2}} \quad \textbf{Cauchy's inequality} \tag{1}$$

$$\sum_{k=1}^{n} a_k b_k \leq \Big(\sum_{k=1}^{n} a_k^p\Big)^{\frac{1}{p}} \Big(\sum_{k=1}^{n} b_k^q\Big)^{\frac{1}{q}} \quad \textbf{Hölder's inequality} \tag{2}$$

where $p > 1$ and $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$

$$\Big(\sum_{k=1}^{n} (a_k + b_k)^p\Big)^{\frac{1}{p}} \leq \Big(\sum_{k=1}^{n} a_k^p\Big)^{\frac{1}{p}} + \Big(\sum_{k=1}^{n} b_k^p\Big)^{\frac{1}{p}} \quad \textbf{Minkowski's inequality} \tag{3}$$

where $p > 1$

$$\sum_{k=1}^{\infty} (a_1 a_2 \dots a_k)^{\frac{1}{k}} < e \sum_{k=1}^{\infty} a_k \quad \textbf{Carleman's inequality} \tag{4}$$

for $a_k > 0$

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_n a_m}{n+m} \leq \pi \Big(\sum_{n=1}^{\infty} a_n^2\Big)^{\frac{1}{2}} \Big(\sum_{m=1}^{\infty} a_m^2\Big)^{\frac{1}{2}} \quad \textbf{Hilbert's inequality} \tag{5}$$

$$\sum_{n=1}^{\infty} \frac{n}{a + 1 + \dots + a_n} \leq 2 \sum_{n=1}^{\infty} \frac{1}{a_n} \quad \textbf{Hardy's inequality} \tag{6}$$

for $a_n > 0$

## 3 Getting acquainted with Carlson's inequality

$$\boxed{\Big(\sum_{k=1}^{\infty} a_k\Big)^4 < \pi^2 \Big(\sum_{k=1}^{\infty} a_k^2\Big) \Big(\sum_{k=1}^{\infty} k^2 a_k^2\Big)}$$

Imagine you are a frisky analysis student (but not so frisky that you have devoured J Michael Steele's wonderful book "The Cauchy-Schwarz Master Class" [1]) who knows about Cauchy's inequality and Hölder's inequality. You now have some hammers and you think you might figuratively nail the proof.

Your first hurdle is the $\pi^2$ factor. This cannot possibly come from the generic $a_k$ themselves since there is in, in general, no assumed trigonometical character, rather it must come from some sort of upper bound on a general sequence. But how? Let that idea fester for a bit.

You think that you might be able to use Cauchy's inequality but the obvious candidates have problems:

$$\begin{aligned}\sum_{k=1}^{n} a_k =& \sum_{k=1}^{n}(a_k.1)\\ \leq& \Big(\sum_{k=1}^{n} a_k^2\Big)^{\frac{1}{2}} \Big(\sum_{k=1}^{n} 1^2\Big)^{\frac{1}{2}}\\ =& \sqrt{n}\Big(\sum_{k=1}^{n} a_k^2\Big)^{\frac{1}{2}}\end{aligned} \tag{7}$$

So $\Big(\sum_{k=1}^{n} a_k)^4 \leq n^2\Big(\sum_{k=1}^{n} a_k^2\Big)^2$ . No cigar.

What about this?

$$\begin{aligned}\sum_{k=1}^{n} a_k =& \sum_{k=1}^{n}(ka_k.\frac{1}{k})\\ \leq& \Big(\sum_{k=1}^{n} k^2a_k^2\Big)^{\frac{1}{2}} \Big(\sum_{k=1}^{n} \frac{1}{k^2}\Big)^{\frac{1}{2}}\\ \leq& \Big(\frac{\pi^2}{6}\Big)^{\frac{1}{2}} \Big(\sum_{k=1}^{n} k^2a_k^2\Big)^{\frac{1}{2}}\end{aligned} \tag{8}$$

So $\Big(\sum_{k=1}^{n} a_k)^4 \leq \Big(\frac{\pi^2}{6}\Big)^2 \Big(\sum_{k=1}^{n} k^2a_k^2\Big)^2$ . No cigar but at least we got $\pi$ into the game thanks to the well known result $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$ .

Not yet crushed the student comes up with an approach a Cockney mathematician would describe as " proper naughty".

$$\begin{aligned}\sum_{k=1}^{n} a_k &= \sum_{k=1}^{n} \frac{a_k}{1+a_k^2}.(1+a_k^2) \\ &\leq \Big(\sum_{k=1}^{n} \frac{a_k^2}{(1+a_k^2)^2}\Big)^{\frac{1}{2}} \Big(\sum_{k=1}^{n} (1+a_k^2)^2\Big)^{\frac{1}{2}} \\ &\leq \Big(\sum_{k=1}^{n} a_k^2\Big)^{\frac{1}{2}} \Big(\sum_{k=1}^{n} k^2(1+a_k^2)^2\Big)^{\frac{1}{2}}\end{aligned} \tag{9}$$

So $\Big(\sum_{k=1}^{n} a_k)^4 \leq \Big(\sum_{k=1}^{n} a_k^2\Big)^2 \Big(\sum_{k=1}^{n} k^2(1+a_k^2)^2\Big)^2$ . Still no cigar .

At this stage the student changes hammers from Cauchy's inequality to Hölder's inequality. As we will see there is still some life in Cauchy's inequality but it requires a bit of magic.

Instead of applying the Hölder hammer to plain old $a_k$ let's modify it a bit:

$$a_k = k^{\frac{-h}{2}} \Big(k^h a_k^2\Big)^{\frac{1}{2}} \tag{10}$$

This doesn't change anything. We will impose a condition on $h$ after we hit $\sum_{k=1}^{n} a_k$ with the Hölder hammer with $p = q = 2$:

$$\begin{aligned}\sum_{k=1}^{n} a_k &= \sum_{k=1}^{n} k^{\frac{-h}{2}} \Big(k^h a_k^2\Big)^{\frac{1}{2}} \\ &\leq \Big(\sum_{k=1}^{n} \Big(k^{\frac{-h}{2}}\Big)^2\Big)^{\frac{1}{2}} \Big(\sum_{k=1}^{n} \Big[\Big(k^h a_k^2\Big)^{\frac{1}{2}}\Big]^2\Big)^{\frac{1}{2}} \\ &= \Big(\sum_{k=1}^{n} k^{-h}\Big)^{\frac{1}{2}} \Big(\sum_{k=1}^{n} k^h a_k^2\Big)^{\frac{1}{2}} \\ &< \Big(\sum_{k=1}^{n} k^{-h}\Big)^{\frac{1}{2}} \Big(\sum_{k=1}^{n} k^2 a_k^2\Big)^{\frac{1}{2}} \text{ assuming we make } 0 < h < 1\end{aligned} \tag{11}$$

Raising the LHS of (11) to the fourth power we get:

$$\Big(\sum_{k=1}^{n} a_k\Big)^4 \leq \Big(\sum_{k=1}^{n} k^{-h}\Big)^2 \Big(\sum_{k=1}^{n} k^2 a_k^2\Big)^2 \tag{12}$$

Once again no cigar but we have managed to get the $\Big(\sum_{k=1}^{n} k^2 a_k^2\Big)^2$ term. If we let $h \to 1$ then $\sum_{k=1}^{n} k^{-h} \to \infty$ (it is just the harmonic series which diverges) which is a problem.

At this stage I rely upon the work of Leo Larsson et al who explain how Carlson "did the business" ( [2] ). I could not find a digital version of the original paper but I assume Larsson et al actually examined it.

But the student has a proper naughty idea. Let's tickle $a_k$ a bit as follows:

$$a_k = k^{\frac{-h}{2}} a_k^{\frac{1}{2}} \Big(k^h a_k\Big)^{\frac{1}{2}} \tag{13}$$

But we now have 3 terms in our new $a_k$. The student has played with the Hölder hammer and has found that if $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$ where $p > 1$, $q > 1$ and $r > 1$ then:

$$\sum_{k=1}^{n} a_k b_k c_k \leq \Big(\sum_{k=1}^{n} a_k^p\Big)^{\frac{1}{p}} \Big(\sum_{k=1}^{n} b_k^q\Big)^{\frac{1}{q}} \Big(\sum_{k=1}^{n} c_k^r\Big)^{\frac{1}{r}} \tag{14}$$

Now in (13) we make the identification $a_k = k^{\frac{-h}{2}}$, $b_k = a_k^{\frac{1}{2}}$ and $c_k = \Big(k^h a_k\Big)^{\frac{1}{2}}$ and we apply (14) with $p = 2$, $q = 4$ and $r = 4$:

$$
\begin{aligned}
\sum_{k=1}^{n} a_k =& \sum_{k=1}^{n} k^{\frac{-h}{2}} a_k^{\frac{1}{2}} \Big(k^h a_k\Big)^{\frac{1}{2}} \\
\leq& \Big(\sum_{k=1}^{n} \Big(k^{\frac{-h}{2}}\Big)^2\Big)^{\frac{1}{2}} \Big(\sum_{k=1}^{n} \Big(a_k^{\frac{1}{2}}\Big)^4\Big)^{\frac{1}{4}} \Big(\sum_{k=1}^{n} \Big(\Big(k^h a_k\Big)^{\frac{1}{2}}\Big)^4\Big)^{\frac{1}{4}} \\
=& \Big(\sum_{k=1}^{n} k^{-h}\Big)^{\frac{1}{2}} \Big(\sum_{k=1}^{n} a_k^2\Big)^{\frac{1}{4}} \Big(\sum_{k=1}^{n} k^{2h} a_k^2\Big)^{\frac{1}{4}} \\
<& \Big(\sum_{k=1}^{n} k^{-h}\Big)^{\frac{1}{2}} \Big(\sum_{k=1}^{n} a_k^2\Big)^{\frac{1}{4}} \Big(\sum_{k=1}^{n} k^2 a_k^2\Big)^{\frac{1}{4}} \text{ since } 0 < h < 1
\end{aligned}
\tag{15}
$$

In [2] at page 10 Larsson et al impose the condition that $h > 1$ but if that were the case then $k^{2h} > k^2$ for $k > 1$, in which case $\sum_{k=1}^{n} k^{2h} a_k^2 > \sum_{k=1}^{n} k^2 a_k^2$. I think this is just a typo and the logic works with $0 < h < 1$ but here is the original so you make your own judgement:

### 1.1 Carlson's Proof

In this section, we briefly sketch Carlson's proof of Theorem 1.1, which is based on the theory of analytic functions. Carlson realized that it is not possible to use the Hölder–Rogers inequality[1] directly to prove (1.1), by arguing as follows: For any $h > 1$, we can write

$$a_k = k^{-\frac{h}{2}} a_k^{\frac{1}{2}} (k^h a_k)^{\frac{1}{2}},$$

and so, by the Hölder–Rogers inequality with exponents $2, 4, 4$, it follows that

$$\begin{aligned}\left(\sum_{k=1}^{\infty} a_k\right)^4 &= \left(\sum_{k=1}^{\infty} k^{-\frac{h}{2}} a_k^{\frac{1}{2}} (k^h a_k)^{\frac{1}{2}}\right)^4 \\ &\leq \left(\sum_{k=1}^{\infty} k^{-h}\right)^2 \sum_{k=1}^{\infty} a_k^2 \sum_{k=1}^{\infty} k^2 a_k^2 \\ &= C(h) \sum_{k=1}^{\infty} a_k^2 \sum_{k=1}^{\infty} k^2 a_k^2,\end{aligned}$$

where

$$C(h) = \left(\sum_{k=1}^{\infty} k^{-h}\right)^2.$$

To get (1.1), one would like to put $h = 1$ in this inequality. However,

$$\lim_{h \searrow 1} C(h) = \infty.$$

It therefore seems as Carlson at this moment regarded his inequality (1.1) as a limiting case which is not covered by the Hölder–Rogers inequality. Hence, it must have been a big surprise for him when G. H. Hardy presented a proof of (1.1) based merely on the Schwarz inequality (see Section 1.2 below).

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Thus, raising (15) to the fourth power we have:

$$\left(\sum_{k=1}^{n} a_k\right)^4 < \left(\sum_{k=1}^{n} k^{-h}\right)^2 \left(\sum_{k=1}^{n} a_k^2\right) \left(\sum_{k=1}^{n} k^2 a_k^2\right) \tag{16}$$

This is nearly Carlson's inequality apart from the factor $\left(\sum_{k=1}^{n} k^{-h}\right)^2$ which, as we have seen in the context of (12) becomes arbitrarily large as $h \to 1$. So close but still no cigar. In fact Carlson went down precisely this route and apparently gave up on the Hölder hammer. Then Hardy came along and proved the inequality only using the Cauchy-Schwarz inequality. Recall that our student gave up on Cauchy-Schwarz so it really does matter who is wielding the hammer! So how did Carlson prove his inequality?

Here is the relevant extract from [2]:

First of all, Carlson showed that (1.1) holds for *some* constant $C$ (in place of $\pi^2$), by using the following argument. Let

$$f(z) = \sum_{k=1}^{\infty} a_k z^k.$$

Then, since $f(0) = 0$, we have

$$\begin{aligned} f(1)^2 &= 2\int_0^1 f(z)f'(z)\,dz \\ &< 2\int_{-1}^1 |f(z)f'(z)|\,dz \\ &\le \int_0^{2\pi} |f(e^{i\varphi})f'(e^{i\varphi})|\,d\varphi \end{aligned}$$

and hence the Schwarz inequality yields

$$\begin{aligned} \left(\sum_{k=1}^{\infty} a_k\right)^4 = f(1)^4 &< \int_0^{2\pi} |f(e^{i\varphi})|^2\,d\varphi \int_0^{2\pi} |f'(e^{i\varphi})|^2\,d\varphi \\ &= 2\pi \sum_{k=1}^{\infty} a_k^2 \cdot 2\pi \sum_{k=1}^{\infty} k^2 a_k^2. \end{aligned}$$

This is (1.1) with $\pi^2$ replaced by $4\pi^2$.

He started by showing that the inequality held for some constant $C$ instead of $\pi^2$. To achieve this he let:

$$f(z) = \sum_{k=1}^{\infty} a_k z^k \tag{17}$$

Note first that $f(0) = 0$ and let $F(z) = f(z)^2$ then:

$$F'(z) = 2f(z)\, f'(z) \tag{18}$$

Integrating between 0 and 1 we have that:

$$\int_0^1 F'(z)dz = 2\int_0^1 f(z)f'(z)\, dz \tag{19}$$

Hence:

$$F(1) - F(0) = f(1)^2 = 2\int_0^1 f(z)f'(z)\, dz \tag{20}$$

So:

$$\begin{aligned} |f(1)^2| \leq & 2\int_0^1 |f(z)f'(z)|\, dz \\ < & 2\int_{-1}^1 |f(z)f'(z)|\, dz \end{aligned} \tag{21}$$

If we assume reasonably that $|f(z)f'(z)|$ is not identically zero on $[-1, 1]$ then adding $\int_{-1}^0 |f(z)f'(z)|\, dz$ is clearly non-negative so the strict inequality applies. The next chain in the logic is that:

$$\boxed{2\int_{-1}^1 |f(z)f'(z)|\, dz \ \leq \int_0^{2\pi} |f(e^{i\varphi})\, f'(e^{i\varphi})|\, d\varphi} \tag{22}$$

This is actually intricate to justify(as you will see below) but Larsson et al clearly assumed a certain level of analytic function theory which they did not bother to even briefly mention.

Once you have (22) Cauchy-Schwarz gives you:

$$\begin{aligned} f(1)^2 &< \int_0^{2\pi} |f(e^{i\varphi})\, f'(e^{i\varphi})|\, d\varphi \\ &< \int_0^{2\pi} |f(e^{i\varphi})|^2\, d\varphi \int_0^{2\pi} |f'(e^{i\varphi})|^2\, d\varphi \end{aligned} \tag{23}$$

Recall that the norm $\|f\|$ is defined by:

$$\|f\|^2 = \frac{1}{2\pi} \int_0^{2\pi} |f(\theta)^2|\, d\theta \tag{24}$$

From Fourier theory we have Parseval's identity:

$$\sum_{n=-\infty}^{\infty} |a_n|^2 = \|f\|^2 \tag{25}$$

Hence:

$$\int_0^{2\pi} |f(e^{i\varphi})|^2\, d\varphi = 2\pi \sum_{k=1}^{\infty} |a_k|^2 \tag{26}$$

noting that the power series for $f(z)$ does not contain any negatives indices. Similarly:

$$\int_0^{2\pi} |f'(e^{i\varphi})|^2\, d\varphi = 2\pi \sum_{k=1}^{\infty} k^2 |a_k|^2 \tag{27}$$

Thus Carlson gets the following:

$$\boxed{\Big(\sum_{k=1}^{\infty} a_k\Big)^4 < 4\pi^2 \sum_{k=1}^{\infty} a_k^2 \sum_{k=1}^{\infty} k^2 a_k^2} \tag{28}$$

which is out by a factor of 4.

## 3.1 An excursion into harmonic and subharmonic theory

Carlson was an expert in analytic function theory and the step in (22) involves some subtle theory. It is not simply a case of making a substitution $z = e^{i\varphi}$ in (21) because a factor of 2 has disappeared and $\varphi = 0 \implies z = 1$ rather than $z = -1$.

Given that $f(z)$ is represented as a power series we know that within its radius of convergence not only is $f(z)$ analytic but so is the derived (ie differentiated ) series. A version of the Maximum Modulus principle assures us that the modulus on the boundary of the disc $|z| = 1$ dominates the functions inside ie:

$$|f(z)| \leq \max_{|w|=1} |f(w)| \text{ for all } |z| \leq 1 \tag{29}$$

In the theory of harmonic functions an important mean value theorem arises. Reference [3] by Eli Stein and Guido Weiss (in particular pages 37-53) covers the major issues). If $u$ is harmonic (ie its second partials are continuous and it satisfies Laplace's equation everywhere in the domain) in a domain $\mathcal{D}$ and if the sphere of radius $r_0$ about $x \in \mathcal{D}$ is contained in $\mathcal{D}$ then for $0 < r \leq r_0$:

$$u(x) = \mathcal{M}_{x,u}(r) = \frac{1}{\omega_{n-1}} \int_{\Sigma} u(x + rt')\, dt' \tag{30}$$

where $\mathcal{M}_{x,u}(r)$ is the mean value , $\Sigma$ is the surface of the unit sphere in $E_n$, $dt'$ is the element of surface area on $\Sigma$ and $\omega_n$ is the surface area. Note that $x = (x_1, x_2, \ldots, x_n)$.

The maximum principle for harmonic functions ( see[3] page 39) states that if $u$ is a real valued harmonic function defined in a domain $\mathcal{D}$ which satisfies $A = \sup_{x \in \mathcal{D}} u(x) < \infty$ , then $u(x) < A$ provided $u$ is not a constant function.

A corollary of the maximum principle is that if $u$ is continuous on the closure $\overline{\mathcal{D}}$ of the bounded domain $\mathcal{D}$ and is harmonic in $\mathcal{D}$ then the maximum (minimum) of $u$ is attained only on the boundary $\partial\mathcal{D} = \overline{\mathcal{D}} - \mathcal{D}$ of $\mathcal{D}$ provided $u$ is not a constant

function (see [3], page 40).

The solution to the Laplace equation in the unit disc subject to the Dirichlet boundary conditions $u(1,\theta) = h(\theta)$ is:

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} h(\phi)\frac{1-r^2}{1-2r\cos(\theta-\phi)+r^2}\,d\phi \tag{31}$$

Note that for $0 \leq r < 1$ the Poisson integral is as follows:

$$P_r(\theta) = \frac{1-r^2}{1-2r\cos\theta+r^2} \tag{32}$$

Sheldon Axler et al ( [5] pages 9-12) sets out some fundamental properties of the Poisson kernel. Thus from the mean value property, if $u$ is harmonic on a ball $\mathcal{B}$ then:

$$u(0) = \int_S u(\xi)\,d\sigma(\xi) \tag{33}$$

where Axler et al are using the Borel probabilty measure $\sigma$ . What Axler et al show is that for every $x \in \mathcal{B}$, $u(x)$ is a weighted average of $u$ over the boundary of $\mathcal{B}$ which the authors denoted by $S$ rather than just $\partial\mathcal{B}$. More precisely they show that there exists a function $P$ on $\mathcal{B} \times S$ such that:

$$u(x) = \int_S u(\xi)\,P(x,\xi)d\sigma(\xi) \tag{34}$$

for every $x \in \mathcal{B}$ and every $u$ harmonic on $\mathcal{B}$. That $P$ is the Poisson kernel.00

To progress the analysis we need the concept of subharmonic functions. If $G$ is a holomorphic function then $|G|^p$ is subharmonic (see [3], page 75) for $p > 0$. For our purposes since $f(z)$ is analytic it is also holomorphic and in the context of (21) we are interested in the subharmonicity of:

$$g(z) = |f(z)\,f'(z)| \tag{35}$$

Since both $f(z)$ and $f'(z)$ are both analytic/holomorphic so is the product of the two and hence $g(z)$ is subharmonic (the absolute value ensures $g(z)$ is positive).

A subharmonic function $s$ is defined as follows ( see [4] page 76): a function $s$ defined and continuous in a domain $\mathcal{D} \subset E_n$ is subharmonic if:

$$s(x_0) \leq \mathcal{M}_{x_0,s}(r) = \frac{1}{\omega_{n-1}} \int_{\Sigma_{n-1}} s(x_0 + rt')\, dt' \tag{36}$$

whenever the sphere $\{x \in E_n : |x - x_0| \leq r\}$ is contained in $\mathcal{D}$. In the case of s single real variable the subharmonic functions are convex functions (see [4], page 75).

In [3] page 80 Stein and Weiss set out an important theorem which I will set out in full (with their symbols):

**Theorem 4.6** Suppose $s$ is a non-negative subharmonic function defined in $E^+_{n+1}$ satisfying

$$\int_{E_n} [s(x,y)]^q\, dx \leq c^q < \infty \tag{37}$$

for all $y > 0$ where $1 \leq q < \infty$ and $c$ is independent of $y > 0$. Then $s$ has a least harmonic majorant in $E^+_{n+1}$. Moreover, in case $q > 1$ this majorant is the Poisson integral of a function $f \in L^q(E_n)$ with $\|f\|_q \leq c$; if $q = 1$, it is the Poisson-Stieltjes integral of a finite Borel measure on $E_n$ having total measure not exceeding $c$.

So going back to (22) we have to use these concepts to show that:

$$\boxed{2 \int_{-1}^{1} |f(z) f'(z)|\, dz \;\leq \int_0^{2\pi} |f(e^{i\varphi})\, f'(e^{i\varphi})|\, d\varphi} \tag{38}$$

Our subharmonic function is $g(z) = |f(z)f'(z)|$ . Using the Maximum Modulus principle for any $z$ we can dominate $g(z)$ by its value on the boundary ie where $r = 1$ and $z = re^{i\varphi}$. Thus:

$$|f(z)f'(z)| \ \leq |f(e^{i\varphi})f'(e^{i\varphi})| \tag{39}$$

But the subharmonic mean property in (36) also gives us:

$$|f(z)f'(z)| \ \leq \frac{1}{2\pi}\int_0^{2\pi} |f(e^{i\varphi})f'(e^{i\varphi})|\, d\varphi \tag{40}$$

Integrating over the diameter $[-1,1]$ we have:

$$\begin{aligned} 2\int_{-1}^{1} |f(z)f'(z)| \ dz \leq & \frac{2}{2\pi}\int_{-1}^{1}\int_0^{2\pi} |f(e^{i\varphi})f'(e^{i\varphi})|\, d\varphi\, dz \\ \leq & \frac{2}{\pi}\int_0^{2\pi} |f(e^{i\varphi})f'(e^{i\varphi})|\, d\varphi \\ < & \int_0^{2\pi} |f(e^{i\varphi})f'(e^{i\varphi})|\, d\varphi \end{aligned} \tag{41}$$

## 4 Refining the constant via minimisation techniques

So at this stage Carlson got to (28) with a factor of $4\pi^2$ rather than $\pi^2$. According to Larsson at al ( [2], pages 11-14) Carlson proceeded as follows. He considered finitely many non-negative numbers $a_k$ for $k = 1, \ldots, m$ which are normalised so that:

$$\sum_{k=1}^{m} a_k = 1 \tag{42}$$

Let

$$R = \sum_{k=1}^{m} a_k^2 \tag{43}$$

Next find the minimum of the following sum:

$$S = \sum_{k=1}^{m} k^2 a_k^2 \tag{44}$$

This minimisation is done subject to the constraints in (42) and (43). Larsson et al get this:

$$a_k = \frac{\lambda}{k^2 + \mu} \tag{45}$$

See this extract:

subject to these constraints. A Lagrange multiplier argument then implies that for some real numbers $\lambda$ and $\mu$, we have

$$a_k = \frac{\lambda}{k^2 + \mu}, \quad k = 1, \ldots, m.$$

The Lagrange multiplier methiod for two constraints with multipliers $\lambda$ and $\mu$ involves solving this (ignoring any Kuhn-Tucker theory surrounding inequalities etc):

$$\begin{aligned} \frac{\partial}{\partial a_k}\left( \sum_{k=1}^{m} k^2 a_k^2 + \lambda\Big( \sum_{k=1}^{m} a_k - 1\Big) + \mu\Big( \sum_{k=1}^{m} a_k^2 - R\Big)\right) &= 0 \\ 2a_k k^2 + \lambda + 2a_k\mu &= 0 \\ \implies a_k &= \frac{-\lambda}{2(k^2+\mu)} \end{aligned} \tag{46}$$

This differs from what Larsson et al got unless we write:

$$A = -\frac{\lambda}{2} \tag{47}$$

where $\lambda \leq 0$ and $k^2 + \mu \geq 0$ to ensure non-negativity of the $a_k$. Thus $A$ is Larsson's $\lambda$.

The next step in the presentation of Larsson et al is to say that the problem is "reduced to finding the minimum of $\sigma$ in the relation:

$$\sum_{k=1}^{m} \frac{1}{(k^2+\mu)^2} \sum_{k=1}^{m} \frac{k^2}{(k^2+\mu)^2} - \sigma \left( \sum_{k=1}^{m} \frac{1}{k^2+\mu} \right)^4 = 0 \tag{48}$$

Going back to (28) and using (43) and (44), (48) is:

$$RS - \sigma \left( \sum_{k=1}^{m} \frac{1}{k^2+\mu} \right)^4 = 0 \tag{49}$$

where the $\sigma$ function encodes the constant. Larsson et al then say that "after a change of variables and a variational argument, Carlson then concludes that the sought constanr $C$ is the reciprocal value of the minimum of the function $\omega$ of $z$, implicitly defined by:

$$\sum_{k=1}^{\infty} \left( \frac{2z^2}{k^2-z^2} \right)^2 \sum_{k=1}^{\infty} k^2 \left( \frac{2z^2}{k^2-z^2} \right)^2 - \omega \left( \sum_{k=1}^{\infty} \frac{2z^2}{k^2-z^2} \right)^4 = 0 \tag{50}$$

where $z$ is a real or purely imaginary number satisfying $z^2 < 1$."

To justify (50) using the scheme of Larsson et al with:

$$a_k = \frac{A}{k^2+\mu} \; k = 1, 2, \ldots, m \tag{51}$$

the constants $A$ and $\mu$ are determined from:

$$\sum_{k=1}^{m} \frac{A}{k^2+\mu} = 1 \tag{52}$$

and

$$\sum_{k=1}^{m} \frac{A^2}{(k^2+\mu)^2} = R \tag{53}$$

If we write:

$$\hat{\sigma} = \inf_{a_k \neq 0} \frac{\left(\sum_{k\geq 1} a_k^2\right)\left(\sum_{k\geq 1} k^2 a_k^2\right)}{\left(\sum_{k\geq 1} a_k\right)^4} \tag{54}$$

then the best constant in the inequality, which is what Carlson was hunting, is:

$$C = \frac{1}{\hat{\sigma}} \tag{55}$$

which gels with the comment by Larsson et al about the "reciprocal of the minimum".

Going back to (49)-(53) we write the following for fixed $m$:

$$A_1(\mu) = \sum_{k=1}^{m} \frac{1}{k^2+\mu} \tag{56}$$

$$A_2(\mu) = \sum_{k=1}^{m} \frac{1}{(k^2+\mu)^2} \tag{57}$$

$$B_2(\mu) = \sum_{k=1}^{m} \frac{k^2}{(k^2+\mu)^2} \tag{58}$$

From (42) and (51) we have that:

$$\begin{aligned} 1 =& \sum_{k=1}^{m} a_k \\ =& \sum_{k=1}^{m} \frac{A}{k^2+\mu} \\ =& A\, A_1(\mu) \\ \implies A =& \frac{1}{A_1(\mu)} \end{aligned} \tag{59}$$

From (43), (56) and (59) we have:

$$\begin{aligned} R =& \sum_{k=1}^{m} a_k^2 \\ =& A^2 \sum_{k=1}^{m} \frac{1}{(k^2+\mu)^2} \\ =& \frac{A_2(\mu)}{A_1(\mu)^2} \end{aligned} \tag{60}$$

From (44) and (58) we have:

$$\begin{aligned} S =& \sum_{k=1}^{m} k^2\, a_k^2 \\ =& A^2 \sum_{k=1}^{m} \frac{k^2}{(k^2+\mu)^2} \\ =& \frac{B_2(\mu)}{A_1(\mu)^2} \end{aligned} \tag{61}$$

Thus:

$$RS = \frac{A_2(\mu)\, B_2(\mu)}{A_1(\mu)^4} \tag{62}$$

and hence we have ( see (49) and (54) ):

$$A_2(\mu)\, B_2(\mu) - \sigma\, A_1(\mu)^4 = 0 \tag{63}$$

We now let $m \to \infty$ and make the substitution:

$$\mu = -z^2 \tag{64}$$

which is the same as (48). Using the substitutiion (64) we have:

$$\frac{1}{k^2 + \mu} = \frac{1}{2z^2} \frac{2z^2}{(k^2 - z^2)} \tag{65}$$

Let:

$$c_k = \frac{2z^2}{k^2 - z^2} \tag{66}$$

Then:

$$\sum_{k=1}^{\infty} \frac{1}{k^2 - z^2} = \frac{1}{2z^2} \sum_{k=1}^{\infty} c_k \tag{67}$$

$$\sum_{k=1}^{\infty} \frac{1}{(k^2 - z^2)^2} = \frac{1}{(2z^2)^2} \sum_{k=1}^{\infty} c_k^2 \tag{68}$$

$$\sum_{k=1}^{\infty} \frac{k^2}{(k^2 - z^2)^2} = \frac{1}{(2z^2)^2} \sum_{k=1}^{\infty} k^2 \, c_k^2 \tag{69}$$

Thus we have:

$$\begin{aligned}
\left( \sum_{k=1}^{\infty} \frac{1}{(k^2 - z^2)^2} \right) \left( \sum_{k=1}^{\infty} \frac{k^2}{(k^2 - z^2)^2} \right) - \omega \left( \sum_{k=1}^{\infty} \frac{1}{k^2 - z^2} \right)^4 &= 0 \\
\frac{1}{(2z^2)^2} \left( \sum_{k=1}^{\infty} c_k^2 \right) \frac{1}{(2z^2)^2} \left( \sum_{k=1}^{\infty} k^2 c_k^2 \right) - \omega \frac{1}{(2z^2)^2} \left( \sum_{k=1}^{\infty} c_k \right)^4 &= 0 \\
\left( \sum_{k=1}^{\infty} c_k^2 \right) \left( \sum_{k=1}^{\infty} k^2 c_k^2 \right) - \omega \left( \sum_{k=1}^{\infty} c_k \right)^4 &= 0 \\
\sum_{k=1}^{\infty} \left( \frac{2z^2}{k^2 - z^2} \right)^2 \sum_{k=1}^{\infty} k^2 \left( \frac{2z^2}{k^2 - z^2} \right)^2 - \omega \left( \sum_{k=1}^{\infty} \frac{2z^2}{k^2 - z^2} \right)^4 &= 0
\end{aligned} \tag{70}$$

which is what Larsson et al arrived at. They then say that after letting $\pi z = u$, these infinite series can be evaluated as follows:

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{2z^2}{k^2 - z^2} &= 1 - u \cot u \\
&= \frac{\sin u - u \cos u}{\sin u}
\end{aligned} \tag{71}$$

Equation (71) is essentially the well known Mittag-Leffler expansion for $\pi \cot(\pi z)$ and of course a Swedish mathematician like Carlson was all over that like a cheap

suit (since Mittag-Leffler was a prominent Swedish mathematician)!

The next step in the Larsson et al exposition is that:

$$\begin{aligned}\sum_{k=1}^{\infty}\left(\frac{2z^2}{k^2-z^2}\right)^2 =& u^2+u^2\,\cot^2 u+u\,\cot u-2\\ =&\frac{u^2+u\,\sin u\cos u-2\sin^2 u}{\sin^2 u}\end{aligned}\tag{72}$$

To prove (72) takes a bit of efforts. We start with (71) in the following form (noting that $\pi z=u$):

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{2z}{k^2-z^2} =&\frac{1}{z}-\frac{u}{z}\cot u\\ =&\frac{1}{z}-\frac{\pi z}{z}\cot \pi z\\ =&\frac{1}{z}-\pi\cot\pi z\end{aligned}\tag{73}$$

If we now differentiate we have that:

$$\begin{aligned}\frac{d}{dz}\left(\frac{2z}{k^2-z^2}\right) =&\frac{2(k^2-z^2)+4z^2}{(k^2-z^2)^2}\\ =&\frac{2(k^2+z^2)}{(k^2-z^2)^2}\end{aligned}\tag{74}$$

Term by term differentiation is kosher (at least when the poles at $z=k$ are avoided) and we have that:

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{2(k^2+z^2)}{(k^2-z^2)^2} =&\frac{d}{dz}\Big(\frac{1}{z}-\pi\cot(\pi z)\Big)\\ =&-\frac{1}{z^2}+\pi^2\csc^2(\pi z)\end{aligned}\tag{75}$$

We can write the following:

$$\left(\frac{2z^2}{k^2-z^2}\right)^2 = \frac{4z^4}{(k^2-z^2)^2} = z^2\,\frac{2(k^2+z^2)}{(k^2-z^2)^2} - \frac{2z^2}{k^2-z^2} \tag{76}$$

Hence we have:

$$\begin{aligned}
\sum_{k=1}^{\infty}\left(\frac{2z^2}{k^2-z^2}\right)^2 =& z^2\sum_{k=1}^{\infty}\frac{2(k^2+z^2)}{(k^2-z^2)^2} - \sum_{k=1}^{\infty}\frac{2z^2}{k^2-z^2}\\
=& z^2\Big(-\frac{1}{z^2} + \pi^2\csc^2(\pi z)\Big) - (1-\pi z\cot(\pi z))\\
=& -1 + z^2\pi^2\csc^2(\pi z) - 1 + \pi z\cot(\pi z)\\
=& z^2\pi^2\csc^2(\pi z) + \pi z\cot(\pi z) - 2\\
=& u^2\csc^2 u + u\cot u - 2 \text{ with } u = \pi z\\
=& u^2(1+\cot^2 u) + u\cot u - 2\\
=& u^2 + u^2\cot^2 u + u\cot u - 2\\
=& u^2 + u^2\frac{\cos^2 u}{\sin^2 u} + u\frac{\cos u}{\sin u} - 2\\
=& \frac{u^2}{\sin^2 u} + \frac{u\sin u\cos u}{\sin^2 u} - \frac{2\sin^2 u}{\sin^2 u}\\
=& \frac{u^2 + u\,\sin u\cos u - 2\sin^2 u}{\sin^2 u}
\end{aligned} \tag{77}$$

as advertised by Larsson et al in (72).

The next thing that Larsson et al assert is that:

$$\begin{aligned}
\sum_{k=1}^{\infty}k^2\left(\frac{2z^2}{k^2-z^2}\right)^2 =& \frac{1}{\pi^2}\Big(u^4 + u^4\cot^2 u - u^3\cot u\Big)\\
=& \frac{1}{\pi^2}\left(\frac{u^4 - u^3\sin u\cos u}{\sin^2 u}\right)
\end{aligned} \tag{78}$$

which can be verified by similar techniques to (77).

Larsson et al then say the following ( the reference to equation (1.2) is a reference to my equation(70) ):

By multiplying through by $\pi^2 \sin^4 u$, (1.2) thus transforms to

$$\pi^2\omega(u\cos u - \sin u)^4 = u^3(u - \sin u\cos u)(u^2 + u\sin u\cos u - 2\sin^2 u).$$

For $u \in (-\pi, \pi)$ or $u$ purely imaginary, it follows from this that $\pi^2\omega > 1$ and $\pi^2\omega \to 1$ as $u$ tends to infinity along the imaginary axis. Thus the non-strict inequality (1.1) holds, and we can also conclude that for any $\epsilon > 0$,



there is a sequence $a_k$ for which

$$\left(\sum_{k=1}^{\infty} a_k\right)^4 > (\pi^2 - \epsilon)\sum_{k=1}^{\infty} a_k^2 \sum_{k=1}^{\infty} k^2 a_k^2.$$

Hence the constant $C = \pi^2$ is sharp. Finally, to show that we in fact have strict inequality in (1.1) for non-zero sequences, Carlson applies the *continuous* version of the inequality

$$\left(\int_0^{\infty} f(x)\,dx\right)^4 \leq \pi^2 \int_0^{\infty} f^2(x)\,dx \int_0^{\infty} x^2 f^2(x)\,dx \qquad (1.3)$$

(to be discussed in Chapter 3) to the function

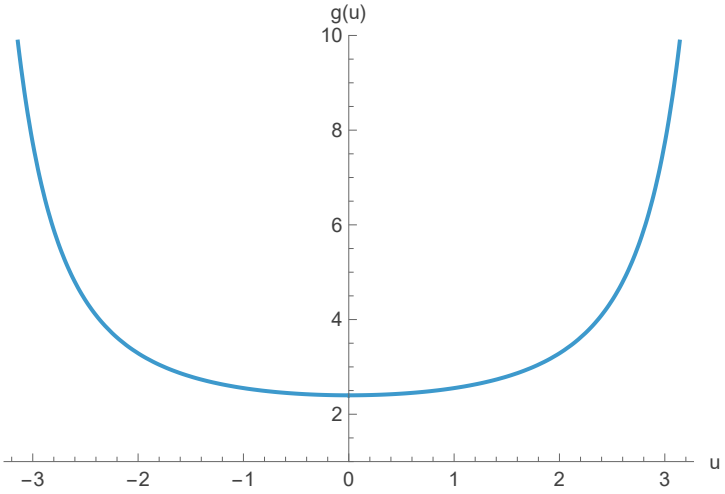
Let us just focus on the claim that for "$u \in (-\pi, \pi)$ or $u$ purely imaginary it follows from this that $\pi^2\omega > 1$":

$$\pi^2\omega\,(u\cos u-\sin u)^4=u^3(u-\sin u\cos u)\,(u^2+u\sin u\cos u-2\sin^2 u) \tag{79}$$

or

$$\pi^2\omega=\frac{u^3(u-\sin u\cos u)\,(u^2+u\sin u\cos u-2\sin^2 u)}{(u\cos u-\sin u)^4} \tag{80}$$

Clearly for $u=0$ (80) suggests that L'Hopital's rule may be useful. Some preliminary numerical exploration is as follows (where $g(u)=\pi^2\omega$):




This suggests that $\pi^2\omega>2$ but we can get a more analytic estimate by the following approximations:

$$\sin u\approx u-\frac{u^3}{6} \tag{81}$$

and

$$\cos u\approx 1-\frac{u^2}{2} \tag{82}$$

It is to be noted that L'Hopital's rule will work with (80) but requires many derivatives because for low order derivatives the denominator is still zero at $u=0$.

When we use the approximations in (81) and (82) and apply L'Hopital's rule once we find that $\pi^2\omega > 3/2 > 1$

```
(* Approximations for sin(u) and cos(u) *)
sinApprox = u - u^3 / 6;
cosApprox = 1 - u^2 / 2;

(* Substitute approximations into g[u] *)
numeratorApprox = u^3 (u - sinApprox cosApprox) (u^2 + u sinApprox cosApprox - 2 sinApprox^2)
denominatorApprox = (sinApprox - u cosApprox) ^4;

(* Compute the simplified ratio *)
simplifiedRatio = Simplify[numeratorApprox / denominatorApprox];

(* Apply L'Hopital's rule *)
numeratorPrime = D[numeratorApprox, u];
denominatorPrime = D[denominatorApprox, u];
limitValue = Limit[numeratorPrime / denominatorPrime, u → 0];

{simplifiedRatio, limitValue}
```

$$\left\{-\frac{3}{16}\left(-8+u^2\right), \frac{3}{2}\right\}$$

Going back to the beginning, the reference in Larsson et al to equation (1.1) is a reference to this:

$$\boxed{\left(\sum_{k=1}^{\infty} a_k\right)^4 < \pi^2\left(\sum_{k=1}^{\infty} a_k^2\right)\left(\sum_{k=1}^{\infty} k^2 a_k^2\right)} \tag{83}$$

and it does follow from (80) with $\pi^2\omega > 1$.

It is an understatement to say that Carlson's proof is intricate. But what is astonishing is that Hardy's proof completely cuts through all the complexity in a way that can only be described as magical.

# 5 Hardy's proof

Hardy's proof assumes that $\alpha$ and $\beta$ are positive real numbers and not all the $a_k$ are 0. The following are defined:

$$S = \sum_{k=1}^{\infty} a_k^2 \tag{84}$$

$$T = \sum_{k=1}^{\infty} k^2 a_k^2 \tag{85}$$

Clearly if either $S$ or $T$ is infinite then there is nothing to prove so we assume they are finite. A simple application of Cauchy-Schwarz yields the following line of inequalities:

$$
\begin{aligned}
\Big(\sum_{k=1}^{\infty} a_k\Big)^2 =& \left(\sum_{k=1}^{\infty} a_k\sqrt{\alpha+\beta k^2}\,\frac{1}{\sqrt{\alpha+\beta k^2}}\right)^2 \\
\leq& \sum_{k=1}^{\infty} a_k^2(\alpha+\beta k^2)\sum_{k=1}^{\infty}\frac{1}{\alpha+\beta k^2} \\
=&(\alpha S+\beta T)\sum_{k=1}^{\infty}\frac{1}{\alpha+\beta k^2} \\
<&(\alpha S+\beta T)\int_0^{\infty}\frac{1}{\alpha+\beta x^2}\,dx \\
=&(\alpha S+\beta T)\,\frac{1}{\alpha}\int_0^{\infty}\frac{1}{1+\left(\sqrt{\frac{\beta}{\alpha}}x\right)^2}\,dx \quad \text{let } u=\sqrt{\frac{\beta}{\alpha}}x \\
=&(\alpha S+\beta T)\,\frac{1}{\sqrt{\alpha\beta}}\int_0^{\infty}\frac{1}{1+u^2}\,du \\
=&(\alpha S+\beta T)\,\frac{1}{\sqrt{\alpha\beta}}\,\frac{\pi}{2} \\
=&\frac{\pi}{2}\Big(\sqrt{\frac{\alpha}{\beta}}S+\sqrt{\frac{\beta}{\alpha}}T\Big) \\
=&\frac{\pi}{2}\Big(\sqrt{\frac{\alpha}{\beta}}\beta+\sqrt{\frac{\beta}{\alpha}}\alpha\Big) \quad \text{with } \alpha=T \text{ and } \beta=S \\
=&\pi\sqrt{\alpha\,\beta}
\end{aligned}
\tag{86}
$$

Thus we have the sought after result:

$$
\Big(\sum_{k=1}^{\infty} a_k\Big)^4 < \pi^2\left(\sum_{k=1}^{\infty} a_k^2\right)\left(\sum_{k=1}^{\infty} k^2 a_k^2\right) \tag{87}
$$

Note that the strict inequality comes from basic integration theory ie dominating $\sum_{k=1}^{\infty}\frac{1}{\alpha+\beta k^2}$ by $\int_0^{\infty}\frac{1}{\alpha+\beta x^2}\,dx$ and this is a strict domination. Here is Hardy's proof from the relevant journal for the record:

A NOTE ON TWO INEQUALITIES

G. H. HARDY*.

1. I give here two very simple proofs of a curious inequality of Carlson, and a supplement to another inequality of Grüss.

Carlson's inequality† is

$$(\Sigma a_n)^4 < \pi^2 \Sigma a_n^2 \Sigma n^2 a_n^2;$$

the sums are from 1 to $\infty$, $a_n \geqslant 0$, and not all $a_n$ are 0. The constant $\pi^2$ is "best possible".

(i) If $\alpha$ and $\beta$ are positive, and

$$S = \Sigma a_n^2, \quad T = \Sigma n^2 a_n^2,$$

then

$$(\Sigma a_n)^2 = \left(\Sigma a_n \sqrt{(\alpha+\beta n^2)} \cdot \frac{1}{\sqrt{(\alpha+\beta n^2)}}\right)^2$$

$$\leqslant \Sigma a_n^2 (\alpha+\beta n^2) \Sigma \frac{1}{\alpha+\beta n^2} = (\alpha S+\beta T) \Sigma \frac{1}{\alpha+\beta n^2}$$

$$< (\alpha S+\beta T) \int_0^\infty \frac{dx}{\alpha+\beta x^2} = \frac{\pi}{2} \left\{S \sqrt{\left(\frac{\alpha}{\beta}\right)} + T \sqrt{\left(\frac{\beta}{\alpha}\right)}\right\}.$$

The result follows on taking $\alpha = T$, $\beta = S$.

* Received 27 January, 1936; read 20 February, 1936.

† F. Carlson, *Arkiv för Mat.*, 25 (1935), B 1.

Remember our frisky analysis student who had the inspiration of equation (9)? That student was so close to Hardy's proof !! He or she shouldn't have given up! It is not uncommon in mathematics for an original proof to be improved up in ways which can be unexpected. I am actually less "convinced" by Carlson's approach than I am by Hardy's.

## 6 References

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# 7 History

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