The summation by parts formula and Dirichlet's convergence test

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1 Background

Dirichlet's convergence test is a very useful tool in analysis to prove convergence of series, especially oscillatory series, the paradigmatic case being:

$$\sum_{n=2}^{\infty} \frac{\sin n}{\ln n} \tag{1}$$

One formulation of Dirichlet's test for convergence is that if we have two real sequences $\{a_n\}$ and $\{b_n\}$ where the partial sums of the series $\sum_n b_n$ are bounded and $\{a_n\}$ is a sequence which monotonically decreases to zero, then $\sum_n a_n b_n$ converges.

This is essentially the form of Dirichlet's test used by the late and great Eli Stein in his textbook of Fourier Analysis ([2], Problem 7, page 60). Stein invites readers first to prove the summation by parts formula and then use it to prove Dirichlet's test for convergence.

2 The summation by parts formula

In Problem 7 referred to above the set up is that $\{a_n\}_{n=1}^{N}$ and $\{b_n\}_{n=1}^{N}$ are finite sequences of complex numbers. Let $B_k = \sum_{n=1}^{k} b_n$ denote the partial sums of the series $\sum_n b_n$ with the convention $B_0 = 0$. Prove that:

$$\sum_{n=M}^{N} a_n b_n = a_N \, B_N - a_M \, B_{M-1} - \sum_{n=M}^{N-1} (a_{n+1} - a_n) \, B_n \tag{2}$$

The structure of (2) is indeed reminiscent of the integration by parts formula. Stein refers to complex series and the summation formula holds for these but in the hypotheses of the Dirichlet test as posed by Stein the a_n are assumed real since for the monotonocity we need an ordered field and when it comes to the proof you will see where this matters. It should be noted that since a complex number is the sum of a real and imaginary component one can separately deal with the two components. The b_k can be complex since when we take their modulus to get the relevant bound you just get a real number. When you track the proof through you will see this.

3 Derivation of the summation by parts formula

The proof can be developed relying upon the simple facts that $b_n = B_n - B_{n-1}$ and the observation that $a_n = a_n - a_{n+1} + a_{n+1}$ and a bit of re-indexing:

$$
\begin{aligned}
\sum_{n=M}^{N} a_n b_n &= \sum_{n=M}^{N} a_n \left(B_n - B_{n-1}\right) \\
&= \sum_{n=M}^{N} \left(a_n - a_{n+1} + a_{n+1}\right)\left(B_n - B_{n-1}\right) \\
&= \sum_{n=M}^{N} \left(a_n - a_{n+1}\right)\left(B_n - B_{n-1}\right) + \sum_{n=M}^{N} a_{n+1}\left(B_n - B_{n-1}\right) \\
&= \sum_{n=M}^{N} \Big(a_n\, B_n - a_n\, B_{n-1} - a_{n+1}\, B_n + a_{n+1}\, B_{n-1}\Big) + \sum_{n=M}^{N} a_{n+1}\, B_n - \sum_{n=M}^{N} a_{n+1}\, B_{n-1} \\
&= a_N\, B_N + \sum_{n=M}^{N-1} a_n\, B_n - a_M\, B_{M-1} - \sum_{n=M+1}^{N} a_n\, B_{n-1} - \cancel{\sum_{n=M}^{N} a_{n+1}\, B} + \cancel{\sum_{n=M}^{N} a_{n+1}\, B_{n-1}} \\
&\quad + \cancel{\sum_{n=M}^{N} a_{n+1}\, B_n} - \cancel{\sum_{n=M}^{N} a_{n+1}\, B_{n-1}} \\
&= a_N\, B_N - a_M\, B_{M-1} + \sum_{n=M}^{N-1} a_n\, B_n - \underbrace{\sum_{n=M+1}^{N} a_n\, B_{n-1}}_{n\mapsto n-1} \\
&= a_N\, B_N - a_M\, B_{M-1} + \sum_{n=M}^{N-1} a_n\, B_n - \sum_{n=M}^{N-1} a_{n+1}\, B_n \\
&= a_N\, B_N - a_M\, B_{M-1} - \sum_{n=M}^{N-1} \left(a_{n+1} - a_n\right) B_n
\end{aligned}
\tag{3}
$$

4 Proof of Dirichlet's convergence test

Recall from the hypotheses of the test that if we have two real sequences $\{a_n\}$ and $\{b_n\}$ where the partial sums of the series $\sum_n b_n$ are bounded and $\{a_n\}$ is a sequence which monotonically decreases to zero, then $\sum_n a_n b_n$ converges.

Since the partial sums $B_n = \sum_k^n b_k$ are bounded this means that $\exists B > 0$ such that $|B_n| = |\sum_k^n b_k| < B$ for all n. This is a "global" bound.

Because $a_n \to 0$, for any $\epsilon > 0$ we can find an index v_1 so large that $|a_n| < \frac{\epsilon}{4B}$ for all $n > v_1$. Similarly because a_n converges it is Cauchy so that we can similarly find an index v_2 such that for all $m > n > v_2$, $|a_{n+1} - a_m| < \frac{\epsilon}{2B}$. Thus if we choose $v > \max\{v_1, v_2\}$ then for $m > n > v$ both estimates hold.

The following proof of Dirichlet's convergence test is based on showing that the relevant partial sum is Cauchy which will guarantee convergence since $a_n \in \mathbb{R}$.

Let $S_n = \sum_{k=1}^n a_k\, b_k$ and take $m > n$ and $\epsilon > 0$.

$$
\begin{aligned}
|S_m - S_n| &= \Big| \sum_{k=1}^{m} a_k\, b_k - \sum_{k=1}^{n} a_k\, b_k \Big| \\
&= \Big| \sum_{k=n+1}^{m} a_k\, b_k \Big| \\
&= \Big| a_m\, B_m - a_{n+1}\, B_n - \sum_{k=n+1}^{m} (a_{k+1} - a_k)\, B_k \Big| \\
&\leq |a_m\, B_m - a_{n+1}\, B_n| + \sum_{k=n+1}^{m} |a_{k+1} - a_k|\, |B_k| \\
&\leq |a_m|\, |B_m| + |a_{n+1}|\, |B_n| + \sum_{k=n+1}^{m} |a_{k+1} - a_k|\, |B_k| \\
&< B\big(|a_m| + |a_{n+1}|\big) + B \sum_{k=n+1}^{m} |a_{k+1} - a_k| \\
&< B\left(\frac{\epsilon}{4B} + \frac{\epsilon}{4B}\right) + B \underbrace{\sum_{k=n+1}^{m} a_k - a_{k+1}}_{\text{using monotonicity}} \\
&= \frac{\epsilon}{2} + B\big(a_{n+1} - \cancel{a_{n+2}} + \cancel{a_{n+2}} - \cancel{a_{n+3}} + \cancel{a_{n+3}} - \cancel{a_{n+4}} + \cdots + \cancel{a_{m-2}} - \cancel{a_{m-1}} + \cancel{a_{m-1}} - a_m\big) \\
&= \frac{\epsilon}{2} + B(a_{n+1} - a_m) \\
&\leq \frac{\epsilon}{2} + \frac{B\epsilon}{2B} \\
&= \epsilon
\end{aligned}
\tag{4}
$$

Thus $|S_m - S_n| < \epsilon$ for all $m > n > v$ showing that S_n is Cauchy and hence is convergent.

5 Application of the convergence test

We can apply Dirichlet's convergence test to the paradigm case in (1) where $b_k = \sin k$ and $a_k = \frac{1}{\ln k}$. Note that since $\ln(k+1) > \ln k$, $a_{k+1} < a_k$ and $a_k \to 0$ since $\ln k \to \infty$. The only issue in applying the test is getting a sensible bound for $B_n = \sum_{k=2}^{n} \sin k$. If we say that $|\sin k| \leq 1$ for all k then $B_n \leq n-1$ which is dependent on n and we need something that is independent of n. This is a common issue with trigonometric series and the way to get a global bound on B_n is to use a telescoping form for $\sin k$ as follows:

$$\sin k = \frac{\cos(k-1) - \cos(k+1)}{2\sin(1)} \tag{5}$$

Hence:

$$\begin{aligned} B_n &= \textstyle\sum_{k=2}^{n} \sin k \\ &= \frac{1}{2\sin(1)}\big(\cos(1) - \cancel{\cos(3)} + \cos(2) - \cancel{\cos(4)} + \cancel{\cos(3)} - \cancel{\cos(5)} + \cdots + \cancel{\cos(n-4)} - \\ &\quad \cancel{\cos(n-2)} + \cancel{\cos(n-3)} - \cancel{\cos(n-1)} + \cancel{\cos(n-2)} - \cos(n) + \cancel{\cos(n-1)} - \cos(n+1)\big) \\ &= \tfrac{1}{2\sin(1)}\big(\cos(1) + \cos(2) - \cos(n) - \cos(n+1)\big). \end{aligned} \tag{6}$$

Taking absolute values:

$$\begin{aligned} |B_n| &= \left|\frac{1}{2\sin(1)}\big(\cos(1) + \cos(2) - \cos(n) - \cos(n+1)\big)\right| \\ &\leq \frac{1}{|2\sin(1)|}\big(|\cos(1)| + |\cos(2)| + |\cos(n)| + |\cos(n+1)|\big) \\ &\leq \frac{4}{|2\sin(1)|} \\ &= \frac{2}{\sin(1)} \end{aligned} \tag{7}$$

So our global bound is $B = \frac{2}{\sin(1)}$. Thus by the Dirichlet convergence test (1) converges.

6 Hardy's proof of the Dirichlet convergence test

Hardy's proof of the test in ([1],pages, 377 and 279) does not explictly rely upon the summation by parts formula, although it uses something similar. He starts by deriving the following result:

If $\phi(n)$ is a positive function of n which tends steadily to zero as $n \to \infty$ then the series: $\phi(0) - \phi(1) + \phi(2) - \ldots$

is convergent and its sum lies between $\phi(0)$ and $\phi(0) - \phi(1)$.

Let $\phi_0 = \phi(0)$, $\phi_1 = \phi(1)$ etc and $s_n = \phi_0 - \phi_1 + \phi_2 - \cdots + (-1)^n \phi_n$

Then $s_{2n+1} - s_{2n-1} = \phi_{2n} - \phi_{2n+1} \geq 0$ and $s_{2n} - s_{2n-2} = -(\phi_{2n-1} - \phi_{2n}) \leq 0$.

Thus $s_0, s_2, s_4, \ldots, s_{2n}, \ldots,$ is a decreasing sequence, and therefore tends to a limit or to $-\infty$ and $s_1, s_3, s_5, \ldots, s_{2n+1}, \ldots$ is an increasing sequence, and therefore tends to a limit or to ∞. But $\lim(s_{2n+1} - s_{2n}) = \lim(-1)^{2n+1}\phi_{2n+1} = 0$, from which it follows that both limits must tend to limits, and that the two limits must be the same. Thus the sequence $s_0, s_1, \ldots, s_n, \ldots$ tends to a limit. Since $s_0 = \phi(0)$, $s_1 = \phi_0 - \phi_1$, it is clear that this limit lies between ϕ_0 and $\phi_0 - \phi_1$.

Hardy states the assumptions of Dirichlet's test this way. If ϕ_n satisfies the conditions mentioned above (ie it is as positive function steadily decreasing to 0 as $n \to \infty$) and $\sum_n a_n$is any series which converges or oscillates finitely, the series $a_0\phi_0 + a_1\phi_1 + a_2\phi_2 + \ldots$ is convergent.

In this set up ϕ_n is like a_n is the earlier proof and the convergent series $\sum_n a_n$ plays the role of the bounded sums B_n in the earlier proof.

Hardy starts out by asserting this:

$$a_0\phi_0 + a_1\phi_1 + \cdots + a_n\phi_n = s_0(\phi_0 - \phi_1) + s_1(\phi_1 - \phi_2) + \cdots + s_{n-1}(\phi_{n-1} - \phi_n) + s_n\phi_n \quad (8)$$

where $s_n = a_0 + a_1 + \cdots + a_n$.

This can be proved by a straight forward induction. The base case for $n = 1$, say, gives $a_0\phi_0 + a_1\phi_1 = s_0(\phi_0 - \phi_1) + s_1\phi_1 = a_0\phi_0 - a_0\phi_1 + a_0\phi_1 + a_1\phi_1$. The induction step is as follows:

$$\begin{aligned} a_0\phi_0 + \cdots + a_n\phi_n + a_{n+1}\phi_{n+1} &= s_0(\phi_0 - \phi_1) + s_1(\phi_1 - \phi_2) + \cdots + s_{n-1}(\phi_{n-1} - \phi_n) + s_n\phi_n + a_{n+1}\phi_{n+1} \\ &= s_0(\phi_0 - \phi_1) + s_1(\phi_1 - \phi_2) + \cdots + s_{n-1}(\phi_{n-1} - \phi_n) + s_n\phi_n + (s_{n+1} - s_n)\phi_{n+1} \\ &= s_0(\phi_0 - \phi_1) + s_1(\phi_1 - \phi_2) + \cdots + s_{n-1}(\phi_{n-1} - \phi_n) + s_n(\phi_n - \phi_{n+1}) + s_{n+1}\phi_{n+1} \end{aligned} \quad (9)$$

Hardy then says the series $(\phi_0 - \phi_1) + (\phi_1 - \phi_2) + \ldots$is convergent , since the sum to n terms is $\phi_0 - \phi_n$ and $\lim \phi_n = 0$. Moreover all its terms are positive. Also since $\sum_n a_n$, if not actually convergent, at any rate oscillates finitely, we can demonstrate a constant K so that $|s_\nu| < K$ for all values of ν. Hence the series $\sum_\nu s_\nu(\phi_\nu - \phi_{\nu+1})$ is absolutely convergent, and so $s_0(\phi_0 - \phi_1) + s_1(\phi_1 - \phi_2) + \cdots + s_{n-1}(\phi_{n-1} - \phi_n)$ tends to a limit as $n \to \infty$. Finally, ϕ_n and therefore $s_n\phi_n$, tends to zero. Hence $a_0\phi_0 + a_1\phi_1 + \cdots + a_n\phi_n$ tends to a limit ie the series $\sum_\nu a_\nu\phi_\nu$ is convergent.

Hardy's style of proof reflects a late 19^{th} century style of analysis that is less common today.

7 References

[1] G H Hardy,, A Course of Pure Mathematics,, Tenth Edition, Cambridge University Press,2006

[2] Elias M Stein and Rami Shakarchi, Fourier Analysis: An Introduction , Princeton Lectures in Analysis 1, Princeton University Press, 2003.

8 History

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