The wave equation and energy conservation

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1 Problem 10, Chapter 3 of "Fourier Analysis: An Introduction" by Elias Stein and Rami Shakarchi

Problem 10 in Chapter 3, page 90, of Elias Stein and Rami Shakarchi's textbook "Fourier Series: An Introduction" [1] contains the following problem which exhibits some fundamental techniques for physics and analysis students.

Here is the problem.

Consider a vibrating string whose displacement $u(x,t)$ at time t satisfies the wave equation:

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \quad c^2 = \frac{\tau}{\rho} \tag{1}$$

ρ is the constant density and τ is the coefficient of tension of the string.

The string is subject to the initial conditions:

$$u(x,0) = f(x) \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = g(x) \tag{2}$$

where it is assumed that $f \in C^1$ and g is continuous. The total energy for the string is defined as:

$$E(t) = \frac{\rho}{2}\int_0^L \big(\frac{\partial u}{\partial t}\big)^2 dx + \frac{\tau}{2}\int_0^L \big(\frac{\partial u}{\partial x}\big)^2 dx \tag{3}$$

(there is a typo in the textbook where T appears instead of τ). The length of the string is L.

The first term in (3) corresponds to the "kinetic energy" of the string (in analogy with $\frac{1}{2}mv^2$, the kinetic energy of a particle of mass m and velocity v), and the second term corresponds to the "potential energy".

Show that the total energy of the string is conserved, in the sense that $E(t)$ is constant. Therefore:

$$E(t) = E(0) = \frac{\rho}{2}\int_0^L g(x)^2\,dx + \frac{\tau}{2}\int_0^L f^{'}(x)^2\,dx \tag{4}$$

2 Detailed solution

The crux of this problem is the definition of total energy in (3) which is merely given, not derived. The kinetic energy component is simply the differential velocity-mass integrated over the length of the string. If the string is displaced a small amount so that $\frac{\partial u}{\partial x}$ is small then the differential arc length $ds = \sqrt{1 + \left(\frac{\partial u}{\partial x}\right)^2}dx$ (this is just Pythagoras' theorem $ds^2 = dx^2 + du^2$). Because $\frac{\partial u}{\partial x}$ is small we can approximate $ds = \sqrt{1 + \left(\frac{\partial u}{\partial x}\right)^2}dx$ by dx and so the mass is simply $\rho\,dx$ and velocity $\frac{\partial u}{\partial t}$ so that the kinetic energy is $\frac{\rho}{2}\int_0^L \left(\frac{\partial u}{\partial t}\right)^2 dx$.

The potential energy is a bit trickier. If we assume that the potential energy is proportional to a differential increase in the length of the string compared with its rest length of L we can get an expression for the potential energy as follows. The factor of proportionality is the tension τ. The change in length is (using Taylor's theorem):

$$\int_0^L \sqrt{1 + \left(\frac{\partial u}{\partial x}\right)^2}dx - L \approx \int_0^L \left(1 + \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2\right)dx - L = \frac{1}{2}\int_0^L \left(\frac{\partial u}{\partial x}\right)^2 dx \tag{5}$$

Accordingly the potential energy is $\frac{\tau}{2}\int_0^L \left(\frac{\partial u}{\partial x}\right)^2 dx$.

There is a more indirect way of arriving at the expression for total energy $E(t)$. Because this involves differentiating under the integral sign, a general theorem is as follows (see [2] pages 268-9):

2.1 Theorem

Let $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that $\frac{\partial g(x,t)}{\partial t}$ exists and is continuous. Suppose $\int_{-\infty}^{\infty} |g(x,t)|\, dx$ and $\int_{-\infty}^{\infty} |\frac{\partial g(x,t)}{\partial t}|\, dx$ and exist for each t and that $\int_{|x|>R} \left| \frac{\partial g(x,t)}{\partial t} \right| dx \to 0$ as $R \to \infty$ uniformly in t on every interval $[a,b]$. Then $\int_{-\infty}^{\infty} g(x,t)\, dx$ is differentiable with:

$\frac{d}{dt} \int_{-\infty}^{\infty} g(x,t)\, dx = \int_{-\infty}^{\infty} \frac{\partial g(x,t)}{\partial t}\, dx$

Let us leave for the moment whether we can actually do the differentiation inside the integral (I will come to the subtleties of that shortly) and simply formally differentiate (4) as follows:

$$E^{'}(t) = \frac{\rho}{2} \int_0^L 2\frac{\partial u}{\partial t} \frac{\partial^2 u}{\partial t^2}\, dx + \frac{\tau}{2} \int_0^L 2\frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial t\, \partial x}\, dx = \rho \int_0^L \frac{\partial u}{\partial t} \frac{\partial^2 u}{\partial t^2}\, dx + \tau \int_0^L \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x\, \partial t}\, dx$$

$$= \rho \int_0^L \frac{\partial u}{\partial t} \frac{\partial^2 u}{\partial t^2}\, dx + \tau \Big[\frac{\partial u}{\partial x} \frac{\partial u}{\partial t} \Big|_0^L - \int_0^L \frac{\partial u}{\partial t} \frac{\partial^2 u}{\partial x^2}\, dx = \rho \int_0^L \frac{\partial u}{\partial t} \frac{\partial^2 u}{\partial t^2}\, dx - \tau \int_0^L \frac{\partial u}{\partial t} \frac{\partial^2 u}{\partial x^2}\, dx$$

$$= \rho\, c^2 \int_0^L \frac{\partial u}{\partial t} \frac{\partial^2 u}{\partial x^2}\, dx - \tau \int_0^L \frac{\partial u}{\partial t} \frac{\partial^2 u}{\partial x^2}\, dx = 0 \quad (6)$$

Note that in equating $\frac{\partial^2 u}{\partial x\, \partial t}\, dx$ with $\frac{\partial^2 u}{\partial t\, \partial x}\, dx$ we are implicitly assuming continuity of the second derivative. In the step involving integration by parts we are assuming initial conditions on the derivatives ie $\frac{\partial u}{\partial t}(0,t) = \frac{\partial u}{\partial t}(L,t) = 0$ for all t. In other words the ends are fixed so that there is no movement and hence no velocity.

Equation (6) shows that $E(t)$ is a constant so that $E(t) = E(0) = \frac{\rho}{2} \int_0^L g(x)^2\, dx + \frac{\tau}{2} \int_0^L f^{'}(x)^2\, dx$ where (2) has been used.

We can derive equation (3) in a more general context by starting with the kinetic energy ie:

$$KE = \frac{\rho}{2} \int_{-\infty}^{\infty} (\frac{\partial u}{\partial t})^2 dx \quad (7)$$

To get convergence of the integral we have to assume that the integrand vanishes outside of some large interval $|x| \leq R$. To see whether the KE is conserved in time we differentiate with respect to time and we get:

$$\frac{d}{dt}KE = \frac{\rho}{2}\int_{-\infty}^{\infty} 2\,\frac{\partial u}{\partial t}\,\frac{\partial^2 u}{\partial t^2}\,dx = \rho\int_{-\infty}^{\infty}\frac{\partial u}{\partial t}\,\frac{\partial^2 u}{\partial t^2}\,dx = \rho\int_{-\infty}^{\infty}\frac{\partial u}{\partial t}\,c^2\,\frac{\partial^2 u}{\partial x^2}dx = \tau\int_{-\infty}^{\infty}\frac{\partial u}{\partial t}\,\frac{\partial^2 u}{\partial x^2}dx \tag{8}$$

The final integral in (8) does not look like it would be zero but if we integrate by parts we come up with something useful:

$$\tau\int_{-\infty}^{\infty}\frac{\partial u}{\partial t}\,\frac{\partial^2 u}{\partial x^2}dx = \tau\,\frac{\partial u}{\partial t}\,\frac{\partial u}{\partial x}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\frac{\partial u}{\partial x}\,\frac{\partial^2 u}{\partial x\,\partial t}\,dx \tag{9}$$

Now $\frac{\partial u}{\partial t}\,\frac{\partial u}{\partial x}\Big|_{-\infty}^{\infty}$ should vanish because the speed of propogation is finite and the derivatives ought to approach zero in the limit. So we are left with:

$$\frac{d}{dt}KE = -\int_{-\infty}^{\infty}\frac{\partial u}{\partial x}\,\frac{\partial^2 u}{\partial x\,\partial t}\,dx = -\frac{d}{dt}\Big(\frac{1}{2}\int_{-\infty}^{\infty}\tau\,\big(\frac{\partial u}{\partial x}\big)^2\,dx\Big) \tag{10}$$

If we define the potential energy to be:

$$PE = \frac{1}{2}\int_{-\infty}^{\infty}\tau\,\big(\frac{\partial u}{\partial x}\big)^2\,dx \tag{11}$$

Then what we have is this:

$$\frac{d}{dt}KE = -\frac{d}{dt}PE \quad \text{or} \quad \frac{d}{dt}(KE + PE) = 0 \tag{12}$$

So (12) says that the total energy defned by the sum of (7) and (11) (which is just (3)) is conserved.

Now let's go right back to basics. If we start with the classic plucked string set up (see [3] pages 39-40) we get the following situation. The string is stretched between $(0,0)$ and $(2,0)$ and the mid-point $(1,0)$ is raised to a height h above the x axis. Thus the initial position of the string is defined by:

$$f(x) = \begin{cases} hx & \text{when} \quad 0 \le x \le 1 \\ -hx + 2h & \text{when} \quad 1 \le x \le 2 \end{cases} \tag{13}$$

The Fourier sine series of (13) has coefficients $b_n = \frac{8h}{\pi^2 n^2}\,\sin\frac{n\pi}{2}$. The formal solution to the wave equation (obtained by separation of variables) $\frac{\partial^2 u}{\partial t^2} = a^2\,\frac{\partial^2 u}{\partial x^2}$ for $0 < x < c$ and $t > 0$ is (see [3] pages 39-40) :

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{c} \cos \frac{n\pi a t}{c} \tag{14}$$

Thus we get the formal solution:

$$u(x,t) = \frac{8h}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin \frac{n\pi}{2} \sin \frac{n\pi x}{2} cos \frac{n\pi a t}{2} \tag{15}$$

which can be shown to be represented as:

$$u(x,t) = \frac{8h}{\pi^2} \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{(2m-1)^2} \sin \frac{(2m-1)\pi x}{2} \cos \frac{(2m-1)\pi a t}{2} \tag{16}$$

Neither (15) or (16) is actually twice differentiable with repect to x or t. To see this look at (15) for instance and note that the damping factor $\frac{8h}{\pi^2} \frac{1}{n^2} \sin \frac{n\pi}{2}$ will be "neutralised" by two differentiations of either the $\sin \frac{n\pi x}{2}$ factor or the $\cos \frac{n\pi a t}{2}$ factor so you will get a pure non-damped oscillatory infinite sum ie something that will not converge. More detailed inforrmation concerning differentiating under the integral sign can be found in [5].

Clearly, then, further conditions are needed to ensure existence and uniqueness of a solution to the wave equation. Churchill provides the proof in Chapter 10 of [3]. He generalizes the problem as follows:

$$\frac{\partial^2 u(x,t)}{\partial t^2} = a^2 \frac{\partial^2 u(x,t)}{\partial x^2} + \phi(x,t), \quad 0 < x < c,\ t > 0 \tag{17}$$

$$u(0,t) = p(t), \quad u(c,t) = q(t) \quad t \geq 0 \tag{18}$$

$$u(x,0) = f(x), \quad \frac{\partial u}{\partial t}(x,0) = g(x) \quad 0 \leq x \leq c \tag{19}$$

It is now assumed that u is of class $\mathcal{C}^2$ in the region $\mathcal{R}$ defined by $0 \leq x \leq c,\ t \geq 0$. Thus u and its first and second derivatives including the mixed second partial derivatives are assumed to be continuous in this region $\mathcal{R}$. Now suppose there are two $\mathcal{C}^2$ solutions $u_1(x,t)$ and $u_2(x,t)$ in the region. Hence the difference $z = u_1 - u_2$ is of class $\mathcal{C}^2$ in the region and it will satisfy the homogeneous problem:

$$\frac{\partial^2 z(x,t)}{\partial t^2} = a^2 \frac{\partial^2 z(x,t)}{\partial x^2}, \quad 0 < x < c, \ t > 0 \tag{20}$$

$$z(0,t) = 0, \quad z(c,t) = 0, \quad t \geq 0 \tag{21}$$

$$z(x,0) = 0, \quad \frac{\partial z}{\partial t}(x,0) = 0, \quad 0 \leq x \leq c \tag{22}$$

We have to show that $z = 0$ throughout the region so that $u_1 = u_2$.

Churchill states ([3], page 240) that "the integrand of the integral:

$$I(t) = \frac{1}{2}\int_0^c (z_x^2 + \frac{1}{a^2} z_t^2)\, dx \tag{23}$$

satisfies conditions such that we can write:

$$I^{'}(t) = \int_0^c (z_x\, z_{xt} + \frac{1}{a^2} z_t\, z_{tt})\, dx\text{''}. \tag{24}$$

Equation (24) ought to look familiar. When c^2 is substitued for a^2, $I(t) = \frac{1}{\tau}\Big(\int_0^c (\frac{\tau}{2}\, z_x^2 + \frac{\rho}{2} z_t^2)\, dx\Big)$ which is $\frac{1}{\tau}\, E(t)$ (taking $z(x,t) = u(x,t)$ and $c = L$).

Now because $z_{tt} = a^2 z_{xx}$, the integrand in (24) becomes:

$$z_x\, z_{xt} + \frac{1}{a^2} z_t\, z_{tt} = z_x\, z_{xt} + z_t\, z_{tt} = \frac{\partial}{\partial x}(z_x\, z_t) \tag{25}$$

Hence

$$I^{'}(t) = \int_0^c \frac{\partial}{\partial x}(z_x\, z_t)\, dx = \Big[z_x(x,t)\, z_t(x,t)\Big]_{x=0}^{x=c} = z_x(c,t)\, z_t(c,t) - z_x(0,t)\, z_t(0,t) = 0 \tag{26}$$

using (22).

Equation (26) shows that $I(t)$ is constant. Now $z(x,0) = 0 \implies z_x(x,0) = 0$ and $z_t(x,0) = 0$ so $I(0) = 0$ and hence $I(t) = 0$. We have a continuous non-negative integrand which is zero at the end-points, hence it must vanish. This is just a special case of a standard result in the calculus of variations (see [4] page 9). We can conclude therefore that $z_x(x,t) = z_t(x,t) = 0$ for $0 \leq x \leq c$ and $t \geq 0$. Thus z must be constant ie $z(x,t) = 0$ because $z(x,0) = 0$.

What this shows that the original boundary value problem set out in (17)-(19) cannot have more than one $\mathcal{C}^2$ solution in $\mathcal{R}$.

Churchill makes the comment ([4]. p.241) that "the requirement of continuity of derivatives of u is severe. Solutions of many simple problems in the wave equation have discontinuities in their derivatives".

Churchill shows ([3[, pages 126-131) how to resolve the problem of the non-convergence of the differentiated series commented upon above.

3 Differentiating under the integral sign

As can be seen from the Theorem, we need continuity of $\frac{\partial g}{\partial t}(x,t)$ where $g(x,t)$ is $\left(\frac{\partial u}{\partial t}\right)^2$ and $\left(\frac{\partial u}{\partial x}\right)^2$ (see (3)). From physical considerations the energy is finite so that we should have existence of $\int_0^L |g(x,t)|\,dx$. As for the integral $\int_{\infty}^{-\infty} |\frac{\partial g}{\partial t}(x,t)|\,dx$, it has to exist for each t. In our case there is a finite spatial interval of integration $[0,L]$ so that given the continuity of the derivatives on this interval there ought to be no blow ups so we can assume this condition is satisfied for all t. The final condition in the theorem requires $\int_{|x|>R} |\frac{\partial g}{\partial t}(x,t)|\,dx \to 0$ as $R \to \infty$ and since we are dealing with a finite spatial interval this should also be satisfied. Hence, differentiating under the integral sign is kosher - certainly good enough to reproduce the derivation for an exam! However, as Churchill pointed out above, it is not at all improbable to see discontinuities in the derivatives in even "simple" wave equation problems so that the analysis becomes much more complex.

4 Energy conservation for the d-dimensional wave equation

The analysis given above for energy conservation of the wave equation in 1 dimension is essentially a bit of a foothold on a much longer and arduous trip to the summit of Mount Everest. In general one would want to establish energy conservation in d dimensions and this is a much more difficult process than for one dimension. Textbooks on partial differential equations (PDEs) usually approach the d-dimensional proof in the context of so-called “energy conservation methods”. There is a variety of approaches some of which omit vast tracts of calculus. In my view one of the best approaches is that of Elias Stein and Rami Shakarchi [1, Chapter 6] because they start off with the 1- dimensional case and then gently show that the solution to a certain wave equation problem (the Cauchy problem) gives rise to energy conservation. Then they leave as an exercise (with hints) the general case. In what follows I retrace what Stein and Shakarchi have done as a necessarily preliminary and then give the proof of the general case in terms of the problem they pose (see Problem 3, [1], pages 213-14)

The wave equation in d dimensions is:

$$\frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_d^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \tag{27}$$

This equation can be rewritten using the Laplacian operator:

$$\Delta u = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \tag{28}$$

where $\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_d^2}$. In what follows we take $c = 1$.

The Cauchy problem is to find a solution to (28) subject to these initial conditions:

$$\left.\begin{array}{rl} u(x,0) = & f(x) \\ \frac{\partial u(x,0)}{\partial t} = & g(x) \end{array}\right\} \tag{29}$$

Both $f, g \in \mathcal{S}(\mathbb{R}^d)$ i.e. they inhabit Schwartz space. Recall that Schwartz space consists of all indefinitely differentiable functions on $\mathbb{R}^d$ such that:

$$\sup_{x\in\mathbb{R}^d} \left| x^\alpha \left(\frac{\partial}{\partial x}\right)^\beta f(x) \right| < \infty \tag{30}$$

for all indices α, β. Hence f, and all its derivatives are rapildy decreasing. See [5.1] for more on Schwartz space. Given that ultimately we want to prove energy conservation in d dimensions we need pretty strong assumptions on the nature of the underlying functions so that we can differentiate under the integral sign and that the integrals at issue actually converge.

5 The Cauchy Problem

Stein and Shakarchi start with the Cauchy problem as follows ([1], pages 185-186). Suppose u solves Cauchy problem described by (28)-(29). We take Fourier transforms of (28) in the space variable as follows by first recalling that:

$$\mathcal{F}\Big(\frac{\partial u(x,t)}{\partial x_k}\Big) \to 2\pi i \xi_k \hat{u}(\xi,t) \tag{31}$$

$\hat{u}(\xi,t)$ is the Fourier transform of $u(x,t)$ with respect to the space variable.

Hence:

$$\mathcal{F}\Big(\frac{\partial^2 u(x,t)}{\partial x_k^2}\Big) \to (2\pi i \xi_k)^2 \hat{u}(\xi,t) = -4\pi\xi_k^2\, \hat{u}(\xi,t) \tag{32}$$

Hence:

$$\mathcal{F}\Big(\Delta u(x,t)\Big) \to -4\pi|\xi|^2\, \hat{u}(\xi,t) \tag{33}$$

The Fourier transform of the RHS of (28) (with c=1) is:

$$\mathcal{F}\Big(\frac{\partial^2 u(x,t)}{\partial t^2}\Big) = \frac{\partial^2 \hat{u}(\xi,t)}{\partial t^2} \tag{34}$$

To see this note that:

$$\mathcal{F}(u(x,t)) = \int_{\mathbb{R}^d} u(x,t)\, e^{-2\pi i x \cdot \xi}\, dx \qquad \text{where } \xi \in \mathbb{R}^d \tag{35}$$

$$\frac{\partial}{\partial t}\,\mathcal{F}(u(x,t)) = \int_{\mathbb{R}^d} \frac{\partial u(x,t)}{\partial t}\, e^{-2\pi i x \cdot \xi}\, dx \qquad \text{where } \xi \in \mathbb{R}^d \tag{36}$$

Note that $x \cdot \xi = \sum_{i=1}^{d} x_i\, \xi_i$.

Then (28) becomes the following Fourier-transformed equation:

$$-4\pi|\xi|^2\, \hat{u}(\xi,t) = \frac{\partial^2 \hat{u}(\xi,t)}{\partial t^2} \tag{37}$$

For each $\xi \in \mathbb{R}^d$, (37) is a manageable ordinary differential equation whose solution is:

$$\hat{u}(\xi,t) = A(\xi)\cos(2\pi\,|\xi| t) + B(\xi)\sin(2\pi\,|\xi| t) \tag{38}$$

Now we have to find $A(\xi)$ and $B(\xi)$ for each ξ and to do this we have to use the initial conditions. We start by taking the Fourier transforms of the initial conditions in (29):

$$\begin{aligned} \mathcal{F}(u(x,0)) &= \mathcal{F}(f(x)) \\ \text{ie } \hat{u}(\xi,0) &= \hat{f}(\xi) = A(\xi) \end{aligned} \tag{39}$$

$$\begin{aligned} \mathcal{F}(\frac{\partial u(x,0)}{\partial t}) &= \mathcal{F}(g(x)) \\ \text{ie } \frac{\partial \hat{u}(\xi,0)}{\partial t} &= \hat{g}(\xi) \end{aligned} \tag{40}$$

Differentiating (38) we thus have from (40) that:

$$\hat{g}(\xi) = 2\pi\,|\xi|\,B(\xi) \implies B(\xi) = \frac{\hat{g}(\xi)}{2\pi\,|\xi|} \tag{41}$$

Thus we have:

$$\hat{u}(\xi,t) = \hat{f}(\xi)\cos(2\pi|\xi|t) + \frac{\hat{g}(\xi)}{2\pi|\xi|}\,\sin(2\pi|\xi|t) \tag{42}$$

Our solution to the Cauchy problem is this:

$$u(x,t) = \int_{\mathbb{R}^d} \left[f(\hat{\xi})\,\cos(2\pi\,|\xi|t) + \frac{\hat{g}(\xi)}{2\pi\,|\xi|}\,\sin(2\pi\,|\xi|t) \right] e^{2\pi i x\cdot\xi}\,d\xi \tag{43}$$

Because $f, g \in \mathcal{S}(\mathbb{R}^d)$ we can differentiate under the integral sign and so u is at least C^2. In fact since both $f, g \in \mathcal{S}(\mathbb{R}^d)$ their Fourier transforms also inhabit Schwartz space (see [1] page 184). So performing the spatial derivative we have for each j:

$$\begin{aligned} \frac{\partial u(x,t)}{\partial x_j} &= \int_{\mathbb{R}^d} \frac{\partial}{\partial x_j}\left[f(\hat{\xi})\,\cos(2\pi\,|\xi|t) + \frac{\hat{g}(\xi)}{2\pi\,|\xi|}\,\sin(2\pi\,|\xi|t) \right] e^{2\pi i x\cdot\xi}\,d\xi \\ &= \int_{\mathbb{R}^d} 2\pi i \xi_j \left[f(\hat{\xi})\,\cos(2\pi\,|\xi|t) + \frac{\hat{g}(\xi)}{2\pi\,|\xi|}\,\sin(2\pi\,|\xi|t) \right] e^{2\pi i x\cdot\xi}\,d\xi \end{aligned} \tag{44}$$

Therefore:

$$
\begin{aligned}
\frac{\partial^2 u(x,t)}{\partial x_j^2} &= \int_{\mathbb{R}^d} 2\pi i \xi_j \, \frac{\partial}{\partial x_j}\Big[\hat{f(\xi)}\, \cos(2\pi\,|\xi| t) + \frac{\hat{g}(\xi)}{2\pi\,|\xi|}\, \sin(2\pi\,|\xi| t)\Big]\, e^{2\pi i x.\xi}\, d\xi \\
&= \int_{\mathbb{R}^d} -4\pi^2 \xi_j^2 \,\Big[\hat{f(\xi)}\, \cos(2\pi\,|\xi| t) + \frac{\hat{g}(\xi)}{2\pi\,|\xi|}\, \sin(2\pi\,|\xi| t)\Big]\, e^{2\pi i x.\xi}\, d\xi
\end{aligned} \tag{45}
$$

Thus the Laplacian is:

$$
\begin{aligned}
\Delta u(x,t) &= \sum_{j=1}^{d} \frac{\partial^2 u(x,t)}{\partial x_j^2} \\
&= \int_{\mathbb{R}^d} \sum_{j=1}^{d} -4\pi^2 \xi_j^2 \,\Big[\hat{f(\xi)}\, \cos(2\pi\,|\xi| t) + \frac{\hat{g}(\xi)}{2\pi\,|\xi|}\, \sin(2\pi\,|\xi| t)\Big]\, e^{2\pi i x.\xi}\, d\xi \\
&= \int_{\mathbb{R}^d} -4\pi^2 |\xi|^2 \,\Big[\hat{f(\xi)}\, \cos(2\pi\,|\xi| t) + \frac{\hat{g}(\xi)}{2\pi\,|\xi|}\, \sin(2\pi\,|\xi| t)\Big]\, e^{2\pi i x.\xi}\, d\xi
\end{aligned} \tag{46}
$$

Now we find $\frac{\partial^2 u(x,t)}{\partial t^2}$:

$$
\begin{aligned}
\frac{\partial u(x,t)}{\partial t} &= \int_{\mathbb{R}^d} \frac{\partial}{\partial t}\Big[\hat{f(\xi)}\, \cos(2\pi\,|\xi| t) + \frac{\hat{g}(\xi)}{2\pi\,|\xi|}\, \sin(2\pi\,|\xi| t)\Big]\, e^{2\pi i x.\xi}\, d\xi \\
&= \int_{\mathbb{R}^d} \Big[-2\pi|\xi|\, \hat{f(\xi)}\, \sin(2\pi\,|\xi| t) + \hat{g}(\xi)\, \cos(2\pi\,|\xi| t)\Big]\, e^{2\pi i x.\xi}\, d\xi
\end{aligned} \tag{47}
$$

$$
\begin{aligned}
\frac{\partial^2 u(x,t)}{\partial t^2} &= \int_{\mathbb{R}^d} \frac{\partial}{\partial t}\Big[-2\pi|\xi|\, \hat{f(\xi)}\, \sin(2\pi\,|\xi| t) + \hat{g}(\xi)\, \cos(2\pi\,|\xi| t)\Big]\, e^{2\pi i x.\xi}\, d\xi \\
&= \int_{\mathbb{R}^d} \Big[-4\pi^2|\xi|^2\, \hat{f(\xi)}\, \cos(2\pi\,|\xi| t) - 2\pi|\xi|\, \hat{g}(\xi)\, \sin(2\pi\,|\xi| t)\Big]\, e^{2\pi i x.\xi}\, d\xi \\
&= \int_{\mathbb{R}^d} -4\pi^2 |\xi|^2 \Big[\hat{f(\xi)}\, \cos(2\pi\,|\xi| t) + \frac{\hat{g}(\xi)}{2\pi|\xi|}\, \sin(2\pi\,|\xi| t)\Big]\, e^{2\pi i x.\xi}\, d\xi
\end{aligned} \tag{48}
$$

This demonstrates that:

$$\frac{\partial^2 u(x,t)}{\partial t^2} = -4\pi^2 |\xi|^2 \, \Delta u(x,t) \tag{49}$$

When $t = 0$ we have:

$$\begin{aligned} u(x,0) &= \int_{\mathbb{R}^d} \hat{f}(\xi)\, e^{2\pi i x \cdot \xi}\, d\xi \\ &= f(x) \quad \text{using the Fourier Inversion Theorem} \end{aligned} \tag{50}$$

Similarly:

$$\begin{aligned} \frac{\partial u(x,0)}{\partial t} &= \int_{\mathbb{R}^d} \hat{g}(\xi)\, e^{2\pi i x \cdot \xi}\, d\xi \\ &= g(x) \quad \text{using the Fourier Inversion Theorem} \end{aligned} \tag{51}$$

Stein and Shakarchi note that since the existence of a solution to the Cauchy problem for the wave equation has been established it is reasonable to wonder about uniqueness. Are there other solutions other than that given above? The answer is "no" and to prove this "energy methods" are employed (see Problem 3, [1], pages 213-14). I will come to that problem shortly but first the authors extend the result of time conservation of total energy of a vibrating string in 1 dimension to higher dimensions. They define energy as follows:

$$\begin{aligned} E(t) &= \int_{\mathbb{R}^d} \Big|\frac{\partial u}{\partial t}\Big|^2 + \Big|\frac{\partial u}{\partial x_1}\Big|^2 + \cdots + \Big|\frac{\partial u}{\partial x_d}\Big|^2 \, dx \\ &= \int_{\mathbb{R}^d} \big|\nabla u(x,t)\big|^2 \, dx \end{aligned} \tag{52}$$

where $\nabla u(x,t) = \begin{pmatrix} \frac{\partial u}{\partial x_1} \\ \vdots \\ \frac{\partial u}{\partial x_d} \\ \\ \frac{\partial u}{\partial t} \end{pmatrix}$

The authors then pose this as a theorem:

Theorem

If u is a solution to the wave equation given by (28) - (29), then E(t) is conserved in the sense that $E(t) = E(0)\ \forall t \in \mathbb{R}$.

Note here that time can be both positive and negative. Unlike the heat equation where time reversal is not possible, the wave equation does admit time reversal.

As a preliminary step in the proof the authors show that if $a, b \in \mathbb{C}$ and $\alpha \in \mathbb{R}$ then:

$$|a \cos\alpha + b \sin\alpha|^2 + |-a\sin\alpha + b\cos\alpha|^2 = |a|^2 + |b|^2 \tag{53}$$

This can be poved by expanding the LHS of (53) or noting that $e_1 = \binom{\cos\alpha}{\sin\alpha}$ and $e_2 = \binom{-\sin\alpha}{\cos\alpha}$ are a pair of orthonormal vectors so that with $Z = \binom{a}{b} \in \mathbb{C}^2$ we have:

$$|Z|^2 = |Z \centerdot e_1|^2 + |Z \centerdot e_2|^2 \tag{54}$$

where " $\centerdot$ " represents the inner product in $\mathbb{C}^2$.

With these preliminairies the authors use Plancherel's formula which states that:

$$\int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 \, d\xi = \int_{\mathbb{R}^d} |f(x)|^2 \, dx \tag{55}$$

so:

$$\begin{aligned}\int_{\mathbb{R}^d} \Big|\frac{\partial u}{\partial t}\Big|^2 dx &= \int_{\mathbb{R}^d} \Big|\frac{\partial}{\partial t}\Big(\hat{f}(\xi)\cos(2\pi|\xi|t) + \frac{\hat{g}(\xi)}{2\pi|\xi|}\ \sin(2\pi|\xi|t)\Big)\Big|^2 d\xi \\ &= \int_{\mathbb{R}^d} |-2\pi|\xi|\ \hat{f(\xi)}\ \sin(2\pi\,|\xi|t) + \hat{g}(\xi)\ \cos(2\pi\,|\xi|t)|^2 \, d\xi\end{aligned} \tag{56}$$

The next relationship is this:

$$\int_{\mathbb{R}^d} \sum_{j=1}^{d} \Big|\frac{\partial u}{\partial x_j}\Big|^2 dx = \int_{\mathbb{R}^d} \Big|2\pi|\xi|\ \hat{f(\xi)}\ \cos(2\pi\,|\xi|t) + \hat{g}(\xi)\ \sin(2\pi\,|\xi\Big|t)|^2 \, d\xi \tag{57}$$

To see this we need to apply Plancherel's theorem and the relationship between a derivative and its Fourier transform (see [1], page 181). Thus we have:

$$\mathcal{F}\Big\{\Big(\frac{\partial}{\partial x}\Big)^{\alpha} u(x,t)\Big\} \to \Big(2\pi i\xi\Big)^{\alpha} \hat{u}(\xi,t) \tag{58}$$

In the present context we have $\alpha = 1$ so that for each spatial coordinate:

$$\mathcal{F}\Big\{\Big(\frac{\partial}{\partial x_j}\Big) u(x,t)\Big\} \to \Big(2\pi i\xi_j\Big) \hat{u}(\xi,t) \tag{59}$$

Hence we have:

$$\mathcal{F}\Big\{\Big|\Big(\frac{\partial}{\partial x_j}\Big) u(x,t)\Big|^2\Big\} \to \Big|2\pi i\xi_j \,\hat{u}(\xi,t)\Big|^2 = \Big|2\pi\xi_j \,\hat{u}(\xi,t)\Big|^2 \tag{60}$$

Now, by Plancherel we have (using (60)):

$$\begin{aligned}\int_{\mathbb{R}^d} \sum_{j=1}^{d} |\frac{\partial u}{\partial x_j}|^2 \, dx &= \int_{\mathbb{R}^d} \sum_{j=1}^{d} \Big|2\pi\xi_j \,\hat{u}(\xi,t)\Big|^2 d\xi \\ &= \int_{\mathbb{R}^d} \sum_{j=1}^{d} \Big|2\pi\xi_j\Big|^2 \Big|\hat{f}(\xi)\cos(2\pi|\xi|t) + \frac{\hat{g}(\xi)}{2\pi|\xi|}\,\sin(2\pi|\xi|t)\Big|^2 d\xi \\ &= \int_{\mathbb{R}^d} \Big(2\pi\,|\xi|\Big)^2 \Big|\hat{f}(\xi)\cos(2\pi|\xi|t) + \frac{\hat{g}(\xi)}{2\pi|\xi|}\,\sin(2\pi|\xi|t)\Big|^2 d\xi \\ &= \int_{\mathbb{R}^d} \Big|2\pi\,|\xi|\,\hat{f}(\xi)\cos(2\pi|\xi|t) + \hat{g}(\xi)\,\sin(2\pi|\xi|t)\Big|^2 d\xi\end{aligned} \tag{61}$$

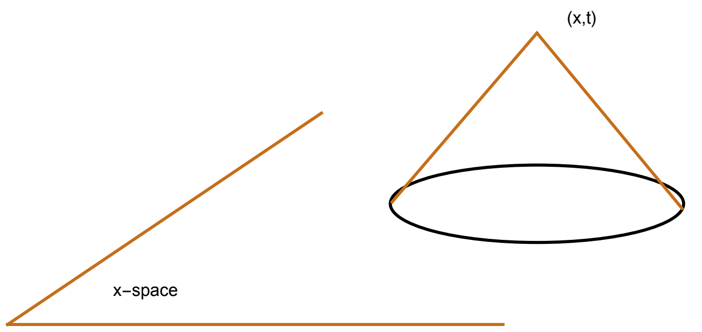
Now using (53) with $a = 2\pi|\xi|\hat{f}(\xi)$, $b = \hat{g}(\xi)$ and $\alpha = 2\pi|\xi|t$ we have:

$$\begin{aligned}E(t) &= \int_{\mathbb{R}^d} |\frac{\partial u}{\partial t}|^2 + |\frac{\partial u}{\partial x_1}|^2 + \cdots + |\frac{\partial u}{\partial x_d}|^2 \, dx \\ &= \int_{\mathbb{R}^d} \Big(4\pi^2|\xi|^2|\hat{f}(\xi)|^2 + |\hat{g}(\xi)|^2\Big)\, d\xi\end{aligned} \tag{62}$$

Since (62) is independent of t and because of (50)-(51), $E(t) = E(0)$ for all t thus proving the claim.

6 Proving uniqueness of the solution to the Cauchy problem using the energy method

Stein and Shakarchi develop the proof of the uniqueness of the solution to the Cauchy problem using the energy method in Problem 3 of Chapter 6 ([1], pages 213-14) as follows. They initially observe that the solution to the wave equation given by (43) depends only on the initial data on the base of the backward light cone. Note that (43) is a highly indirect solution. In this context the authors say ([1], page 196) that " the solution to at a point (x,t) depends only on the data at the base of the backward light cone originating at (x,t). In fact when $d > 1$ is odd, only the data in an immediate neighborhood of the boundary of the base will affect $u(x,t)$".



Backward light cone originating at (x,t)

They then go on to ask if this property is shared by any solution of the wave equation with the implication of an affirmative answer being that the solution is unique. They define $B(x_0, r_0)$ to be the closed ball in the hyperplane $T = 0$ centred at x_0 and of radius r_0. The backward light cone with base $B(x_0, r_0)$ is then defined to be:

$$\mathcal{L}_{B(x_0,r_0)} = \{(x,t) \in \mathbb{R}^d \times \mathbb{R} : |x - x_0| \leq r_0 - t,\ 0 \leq t \leq r_0\} \tag{63}$$

More information on the concept of a light cone and the differences between the heat and wave equations in terms of propagation speeds can be found in ([7], pages 113-115).

Thus Problem 3 becomes the proof of the following theorem:

Theorem Suppose that $u(x,t)$ is a $\mathcal{C}^2$ function on the closed upper half-plane $\{(x,t) : x \in \mathbb{R}^d, t \geq 0\}$ that solves the wave equation $\frac{\partial^2 u}{\partial t^2} = \Delta u$. If $u(x,0) = \frac{\partial u}{\partial t}(x,0) = 0$ for all $x \in B(x_0, r_0)$, then $u(x,t) = 0$ for all $(x,t) \in \mathcal{L}_{B(x_0,r_0)}$.

If the initial data of the Cauchy problem for the wave equation vanishes on a ball B, then any solution of the problem vanishes in the backward light cone with base B.

The steps in the proof are as follows. It is assumed that u is real and for $0 \leq t \leq r_0$ let $B_t(x_0, r_0) = \{x : |x_-x_0| \leq r_0 - t\}$ and $\nabla u(x,t) = \begin{pmatrix} \frac{\partial u}{\partial x_1} \\ \vdots \\ \frac{\partial u}{\partial x_d} \\ \\ \frac{\partial u}{\partial t} \end{pmatrix}$.

The energy integral is given by:

$$\begin{aligned} E(t) =& \frac{1}{2}\int_{B_t(x_0,r_0)} \left|\nabla u(x,t)\right|^2 dx \\ =& \frac{1}{2}\int_{B_t(x_0,r_0)} \left\{ \left(\frac{\partial u}{\partial t}\right)^2 + \sum_{j=1}^{d} \left(\frac{\partial u}{\partial x_j}\right)^2 \right\} dx \end{aligned} \tag{64}$$

Clearly, from (64) $E(t) \geq 0$. Also $E(0) = 0$ because of the following. First, from the initial conditions $\frac{\partial u}{\partial t} u(x,0) = 0$ for all $x \in B(x_0, r_0)$ so $\left(\frac{\partial u}{\partial t}\right)^2 = 0$ in the ball $B_0(x_0, r_0)$. Second, we have:

$$\begin{aligned} \frac{\partial u}{\partial t} u(x,0) =& \frac{\partial u}{\partial x_j} u(x,0)\, \frac{\partial x_j}{\partial t} \\ 0 =& \frac{\partial u}{\partial x_j} u(x,0)\, \times \underbrace{\frac{\partial x_j}{\partial t}}_{\text{this is not indentically zero for all j since } c=1} \end{aligned} \tag{65}$$

Hence since the LHS of (65) is zero for all $x \in B_0(x_0, r_0)$ we must have that $\frac{\partial}{\partial x_j} u(x,0) = 0$ for all j. Thus $\sum_{j=1}^{d} \left(\frac{\partial u}{\partial x_j}\right)^2 = 0$ and so $E(0) = 0$.

The authors invite you to prove the following:

$$E^{'}(t) = \int_{B_t(x_0,r_0)} \left\{ \frac{\partial u}{\partial t}\,\frac{\partial^2 u}{\partial t^2} + \sum_{j=1}^{d} \frac{\partial u}{\partial x_j}\,\frac{\partial^2 u}{\partial x_j\,\partial t} \right\} dx - \frac{1}{2}\int_{\partial B_t(x_0,r_0)} \left|\nabla u(x,t)\right|^2 d\sigma(\gamma) \tag{66}$$

Note that $d\sigma(\gamma)$ denotes the surface element on the sphere $\mathcal{S}^{d-1}$ using spherical coordinates.

This is a version of the Reynolds Transport Equation which, in the context of fluid dynamics, generalises differentiation under the integral sign (Leibniz's Rule) with a moving boundary. Harley Flanders has given a "physicist's proof" of the general formula ([6]) which I essentially reproduce below in the Appendix. There are very few proofs, let alone good ones, of this formula in undergraudate mathematical textbooks. There is a deep cross-over with the theory of partial derivatives in the context of harmonic functions and hence there is a substantial overhead in getting one's mind around the detail. One can find proofs in physics textbooks such as [8] or partial differential equation textbooks such as [7] and [10]. It is common to find short proofs in partial differentiation textbooks (especially those pitched at graduate level) which assume a whole edifice of knowledge about harmonic functions. In fact in his article over 40 years ago Flanders said this ([6.1], page 616):

"We shall discuss generalizations of the Leibniz rule to more than one dimension. Such generalizations seem to be common knowledge among physicists, some differential geometers, and applied mathematicians who work in continuum mechanics,, but are virtually unheard of among most mathematicians. I cannot find a single mention of such formulas in the current advanced calculus and several variable texts, except for Loomis and Sternberg".

Not much has changed since 1973.

Rather than go into what Flanders did here, I have put the detail in the Appendix which you should read if you want to understand where the Reynolds Transport Equation comes from, at least as a mathematical exercise (he was not trying to give the sort of proof one finds in fluid mechanics textbooks). There is a lot of detail (I have expanded on Flanders' treatment) because there is no short way to understand where the formula comes from. The Reynolds Transport Equation (RTE) can be researched at a high level in Wikipedia: `https://en.wikipedia.org/wiki/Leibniz_integral_rule`

The RTE can be expressed as follows:

$$\boxed{\frac{d}{dt}\int_{D_t} F(\vec{x},t)\,dV = \int_{D_t} \frac{\partial}{\partial t}F(\vec{x},t)\,dV + \int_{\partial D_t} F(\vec{x},t)\vec{v_b} \cdot \hat{n}\,dS} \tag{67}$$

Here $F(\vec{x},t)$ is a scalar function and D_t and ∂D_t denote time varying connected regions and its boundary which moves at velocity $\vec{v_b}$ and $\hat{n}$ is the unit normal component of the surface element.

Once you know the RTE it is easy to prove (66). From (64):

$$F(\vec{x},t) = \frac{1}{2}\Big\{\Big(\frac{\partial u}{\partial t}\Big)^2 + \sum_{j=1}^{d}\Big(\frac{\partial u}{\partial x_j}\Big)^2\Big\} \tag{68}$$

Hence the first integral on the RHS of (67) is:

$$\begin{aligned}
\int_{B_t(x_0,r_0)} \frac{\partial}{\partial t}F(\vec{x},t)\,dV &= \int_{B_t(x_0,r_0)} \frac{\partial}{\partial t}\frac{1}{2}\Big\{\Big(\frac{\partial u}{\partial t}\Big)^2 + \sum_{j=1}^{d}\Big(\frac{\partial u}{\partial x_j}\Big)^2\Big\}\,dV \\
&= \int_{B_t(x_0,r_0)} \Big\{\Big(\frac{\partial u}{\partial t}\Big)\frac{\partial^2 u}{\partial t^2} + \sum_{j=1}^{d}\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_j \partial t}\Big\}\,dV \\
&= \int_{B_t(x_0,r_0)} \Big\{\Big(\frac{\partial u}{\partial t}\Big)\frac{\partial^2 u}{\partial t^2} + \sum_{j=1}^{d}\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_j \partial t}\Big\}\,dx
\end{aligned} \tag{69}$$

Note that the authors use dx to mean $dV = dx_1 dx_2 \ldots dx_d$.

To work out the second integral on the RHS of (67) we have to remember that the velocity of the boundary, $\vec{v_b}$ is the velocity of the base of the light cone which we have normalised to $c = 1$ so , $\vec{v_b} = \vec{1}$. The next thing to note is that the unit normal has a negative sign (see the diagram of the light cone). Thus recalling (64) we have:

$$\int_{\partial B_t(x_0,r_0)} F(\vec{x},t)\vec{v_b} \cdot \hat{n}\,dS = -\frac{1}{2}\int_{\partial B_t(x_0,r_0)} \big|\nabla u(x,t)\big|^2\,d\sigma(\gamma) \tag{70}$$

Putting (69)-(70) together we do get (66) ie:

$$E^{'}(t) = \int_{B_t(x_0,r_0)} \Big\{\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2} + \sum_{j=1}^{d}\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_j\,\partial t}\Big\}\,dx - \frac{1}{2}\int_{\partial B_t(x_0,r_0)} \big|\nabla u(x,t)\big|^2\,d\sigma(\gamma) \tag{71}$$

The next step in the proof is to note that:

$$\underbrace{\frac{\partial}{\partial x_j}\Big[\frac{\partial u}{\partial x_j}\,\frac{\partial u}{\partial t}\Big]}_{\frac{\partial \Phi_i}{\partial x_j}} = \frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_j \partial t} + \frac{\partial^2 u}{\partial x_j^2}\frac{\partial u}{\partial t} \tag{72}$$

Using (71) and the divergence theorem and the fact that u solves the wave equation we have to show that:

$$E^{'}(t) = \int_{\partial B_t(x_0,r_0)} \sum_{j=1}^{d} \frac{\partial u}{\partial x_j}\frac{\partial u}{\partial t}\,\nu_j\, d\sigma(\gamma) - \frac{1}{2}\int_{\partial B_t(x_0,r_0)} \left|\nabla u(x,t)\right|^2 d\sigma(\gamma) \tag{73}$$

where v_j is the j^{th} coordinate of the outward normal to $B_t(x_0, r_0)$. At this point recall that the Divergence Theorem says this for 3-space and more generally for d dimensions with appropriate modifications:

$$\iiint\limits_{\mathcal{D}} \nabla \cdot \vec{F}\, dV = \iint\limits_{\partial\mathcal{D}} \vec{F} \cdot \vec{n}\, dS \tag{74}$$

where $\vec{n}$ is the outward unit normal to the surface.

Using (72) in (71) we have:

$$\begin{aligned}
E^{'}(t) &= \int_{B_t(x_0,r_0)} \left\{ \frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2} + \sum_{j=1}^{d}\left\{ \frac{\partial \Phi_j}{\partial x_j} - \frac{\partial^2 u}{\partial x_j^2}\frac{\partial u}{\partial t}\right\}\right\} dx - \frac{1}{2}\int_{\partial B_t(x_0,r_0)} \left|\nabla u(x,t)\right|^2 d\sigma(\gamma) \\
&= \int_{B_t(x_0,r_0)} \left\{ \frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2} + \nabla \cdot \vec{\Phi} - \Delta u \frac{\partial u}{\partial t}\right\} dx - \frac{1}{2}\int_{\partial B_t(x_0,r_0)} \left|\nabla u(x,t)\right|^2 d\sigma(\gamma) \\
&= \int_{B_t(x_0,r_0)} \underbrace{\left\{ \frac{\partial^2 u}{\partial t^2} - \Delta u\right\}}_{=0} \frac{\partial u}{\partial t}\, dx + \int_{B_t(x_0,r_0)} \nabla \cdot \vec{\Phi}\, dx - \frac{1}{2}\int_{\partial B_t(x_0,r_0)} \left|\nabla u(x,t)\right|^2 d\sigma(\gamma) \\
&= \int_{B_t(x_0,r_0)} \nabla \cdot \vec{\Phi}\, dx - \frac{1}{2}\int_{\partial B_t(x_0,r_0)} \left|\nabla u(x,t)\right|^2 d\sigma(\gamma) \\
&= \underbrace{\int_{\partial B_t(x_0,r_0)} \vec{\Phi} \cdot \vec{n}\, d\sigma(\gamma)}_{\text{Divergence Theorem}} - \frac{1}{2}\int_{\partial B_t(x_0,r_0)} \left|\nabla u(x,t)\right|^2 d\sigma(\gamma) \\
&= \int_{\partial B_t(x_0,r_0)} \sum_{j=1}^{d} \frac{\partial u}{\partial x_j}\frac{\partial u}{\partial t}\, v_j\, d\sigma(\gamma) - \frac{1}{2}\int_{\partial B_t(x_0,r_0)} \left|\nabla u(x,t)\right|^2 d\sigma(\gamma)
\end{aligned} \tag{75}$$

To get a bound on $\sum_{j=1}^{d} \frac{\partial u}{\partial x_j}\frac{\partial u}{\partial t}\, v_j$ where v_j is the j^{th} coordinate of the unit outward normal to $B_t(x_0, r_0)$ we note that:

$$xy \leq \frac{1}{2}x^2 + \frac{1}{2}y^2 \quad \forall x, y \in \mathbb{R} \tag{76}$$

Letting $x = a_k$ and $y = b_k$ we then have:

$$\sum_{j=1}^{d} a_k b_k \leq \frac{1}{2}\sum_{j=1}^{d} a_k^2 + \frac{1}{2}\sum_{j=1}^{d} b_k^2 \tag{77}$$

(77) is a species of a Hölder inequality (see [9], pages 136-137).

Hence:

$$\begin{aligned}
\sum_{j=1}^{d} \underbrace{\frac{\partial u}{\partial x_j}}_{a_j} \underbrace{\frac{\partial u}{\partial t} v_j}_{b_j} \leq \frac{1}{2}\sum_{j=1}^{d}\left(\frac{\partial u}{\partial x_j}\right)^2 + \frac{1}{2}\sum_{j=1}^{d}\left(\frac{\partial u}{\partial t}\right)^2 v_j^2 \\
=& \frac{1}{2}\sum_{j=1}^{d}\left(\frac{\partial u}{\partial x_j}\right)^2 + \frac{1}{2}\left(\frac{\partial u}{\partial t}\right)^2 \underbrace{\sum_{j=1}^{d} v_j^2}_{=1} \\
=& \frac{1}{2}\sum_{j=1}^{d}\left\{\left(\frac{\partial u}{\partial x_j}\right)^2 + \left(\frac{\partial u}{\partial t}\right)^2\right\} \\
=& \frac{1}{2}|\nabla u(x,t)|^2
\end{aligned} \tag{78}$$

Thus going back to (75) we have:

$$\begin{aligned}
E^{'}(t) =& \int_{\partial B_t(x_0,r_0)} \sum_{j=1}^{d} \frac{\partial u}{\partial x_j}\frac{\partial u}{\partial t} v_j \, d\sigma(\gamma) - \frac{1}{2}\int_{\partial B_t(x_0,r_0)} \left|\nabla u(x,t)\right|^2 d\sigma(\gamma) \\
\leq& \frac{1}{2}\int_{\partial B_t(x_0,r_0)} |\nabla u(x,t)|^2 \, d\sigma(\gamma) - \frac{1}{2}\int_{\partial B_t(x_0,r_0)} |\nabla u(x,t)|^2 \, d\sigma(\gamma) \\
=& 0
\end{aligned} \tag{79}$$

Thus we have that $E^{'}(t) \leq 0$ and from (64)-(65) we also know that $E(t) \geq 0$ for all $t \geq 0$ and $E(0) = 0$. From the definition of the derivative we have for $h > 0$:

$$
\begin{aligned}
E^{'}(0) = \lim_{h \to 0^+} &= \frac{E(0+h) - E(0)}{h} \\
&= \lim_{h \to 0^+} \frac{E(h)}{h} \leq 0
\end{aligned}
\tag{80}
$$

But $E(h) \geq 0$ so we have a contradiction and so $E(t) = 0$ for all $t \geq 0$. Alternatively the negative derivative implies that the energy is a decreasing function of increasing time but $E(t) \geq 0$ and $E(0) = 0$ so energy must always be zero.

Also because $u(x, 0) = \frac{\partial u(x,0)}{\partial t} = 0$ for all $x \in B(x_0, r_0)$ we have that $u = 0$ since the terms in the expression for $E(t)$ in (64) must be zero.

7 Appendix

Flanders starts out by noting that if:

$$
\Phi(u, v, t) = \int_u^v F(x, t)\, dx \tag{81}
$$

where $u = g(t)$ and $v = h(t)$ the chain rule gives:

$$
\frac{d}{dt}\Phi[g(t), h(t), t] = \underbrace{\left(\frac{\partial \Phi}{\partial u}\,\dot{g} + \frac{\partial \Phi}{\partial v}\,\dot{h} \right)}_{\text{interval variation term}} + \frac{\partial \Phi}{\partial t} \tag{82}
$$

The bracketed terms measure changes due to the variation in the interval of integration $[g(t), h(t)]$ while the remaining term measure changes dues to the variation of the integrand. Assuming sufficient smoothness to justify interchange of integration and differentation operators we have:

$$
\frac{\partial \Phi}{\partial t} = \frac{\partial}{\partial t} \int_u^v F(x, t)\, dx = \int_u^v \frac{\partial F(x, t)}{\partial t}\, dx \tag{83}
$$

To deal with the multidimensional case the problem is with the moving domain rather than the varying integrand but (82) shows us how to separate the two components. So Flanders starts with this:

$$\frac{d}{dt}\int_{g(t)}^{h(t)} F(x)\,dx \tag{84}$$

The domain of integration is an interval $C_t = [g(t), h(t)]$ which is moving in time but what happens in the interior of the domain is not clear. What Flanders does is to imagine that the interval C_t is a worm crawling along the x-axis so that as it stretches and shrinks each point of its body can only shrink so much, so $\frac{\partial x}{\partial u} > 0$. Thus for each t, the map $u \to x(u,t)$ is smooth one-one with smooth (continuously differentiable) inverse. He then writes:

$$\begin{aligned} \phi_t(u) =& x(u,t) \\ \phi_t : [a,b] \to [\phi_t(a), \phi_t(b)] =& [g(t), h(t)] = C_t \end{aligned} \tag{85}$$

Thus by changing the variable in an integral we have:

$$\int_{g(t)}^{h(t)} F(x)\,dx = \int_{\phi_t(a)}^{\phi_t(b)} F(x)\,dx = \int_a^b F[x(u,t)]\,\frac{\partial x}{\partial u}\,du \tag{86}$$

The significance of this is that moving domain is now fixed but the integrand is now time-varying.

Flanders now differentiates under the integral sign:

$$\begin{aligned} \frac{d}{dt}\int_{g(t)}^{h(t)} F(x)\,dx =& \frac{d}{dt}\int_a^b F[x(u,t)]\,\frac{\partial x}{\partial u}\,du \\ =& \int_a^b \frac{\partial}{\partial t}\Big\{F[x(u,t)]\,\frac{\partial x}{\partial u}\Big\}\,du \\ =& \int_a^b \Big\{F^{'}[x(u,t)]\,\frac{\partial x}{\partial t}\frac{\partial x}{\partial u} + F[x(u,t)]\,\frac{\partial^2 x}{\partial u\,\partial t}\Big\}\,du \end{aligned} \tag{87}$$

Flanders goes on to explain that the focus is now on the moving domain whose instantaneous velocity is $v = v(u,t) = \frac{\partial x}{\partial t}$ which he treats as function of x and t via the transformation $(u,t) \leftrightarrow (x,t)$. Thus when t is fixed:

$$\frac{\partial^2 x}{\partial u\,\partial t} = \frac{\partial v}{\partial u} = \left(\frac{\frac{\partial v}{\partial u}}{\frac{\partial x}{\partial u}}\right)\frac{\partial x}{\partial u} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial u} \tag{88}$$

Hence (with t fixed throughout):

$$
\begin{aligned}
\frac{d}{dt}\int_{g(t)}^{h(t)} F(x)\,dx &= \int_a^b \left\{F^{'}[x(u,t)]\,\frac{\partial u}{\partial t}\,\frac{\partial x}{\partial u} + F[x(u,t)]\,\frac{\partial^2 x}{\partial u \partial t}\,du\right\} \\
&= \int_{\Phi_t(a)}^{\Phi_t(b)} \left\{F^{'}(x)\,v\,\frac{\partial x}{\partial u} + F(x)\,\frac{\partial v}{\partial x}\frac{\partial x}{\partial u}\right\}du \\
&= \int_{\Phi_t(a)}^{\Phi_t(b)} \left\{F^{'}(x)\,v\, + F(x)\,\frac{\partial v}{\partial x}\right\}dx \\
&= \int_{\Phi_t(a)}^{\Phi_t(b)} \frac{\partial}{\partial x}[vF(x)]\,dx \\
&= \int_{g(x)}^{h(x)} \frac{\partial}{\partial x}[vF(x)]\,dx
\end{aligned}
\tag{89}
$$

Flanders notes that the derivative has been expressed as an integral over a moving domain and the integrand depends on the velocity v at each point of the domain. However, the integrand is an exact derivative so the answer depends only on the boundary values. At the boundary points $g(t)$ and $h(t)$ the respective velocities are $\dot{g}(t)$ and $\dot{h}(t)$ so that:

$$
\begin{aligned}
\frac{d}{dt}\int_{g(t)}^{h(t)} F(x)\,dx &= \int_{g(t)}^{h(t)} \frac{\partial}{\partial x}[vF(x)]\,dx \\
&= \Bigg[vF(x)\Bigg]_{g(t)}^{h(t)} \\
&= F[h(t)]\,\dot{h}(t) - F[g(t)]\,\dot{g}(t)
\end{aligned}
\tag{90}
$$

Flanders concedes that this may be a silly approach because of the introduction of the unnecessary quantity v and by side-stepping the use of the fundamental theorem initially only to use it in the end anyway. However, the ultimate goal was the reduction to a fixed domain. Next Flanders imagines a moving domain D_t (see Figure 1) in the $x - y$ plane. Flanders proceeds by separating the boundary variation from the integrand variation.

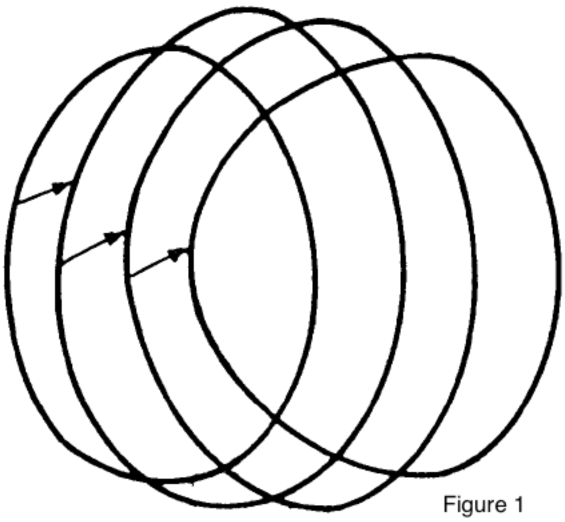


Figure 1

He fixes $t = t_0$ and uses the chain rule as before (see (82)). Thus we have:

$$\frac{d}{dt}\iint_{D_t} F(x,y,t)\,dx\,dy\bigg|_{t=t_0} = \frac{d}{dt}\iint_{D_t} F(x,y,t_0)\,dx\,dy\bigg|_{t=t_0} + \iint_{D_{t_0}} \frac{\partial F}{\partial t}\bigg|_{t=t_0} dx\,dy \tag{91}$$

To work out the LHS of (91) Flanders uses what he calls a "physicist's argument". All this means is that the argument doesn't have epsilons and deltas in it ! The logic of the derivation is common to both disciplines. It is how I remember such proofs and if you are pushed you dot all the "i's" and cross all the "t's". He starts by considering two successive domains as shown in Figure 2. If we let $\mathbf{v} = \mathbf{v}(x,y,t)$ denote the velocity at a boundary point (x,y) of D_t and let $\mathbf{n}$ denote the outward unit normal in the following difference everything in the overlap of D_t and D_{t+dt} cancels so that only a thin boundary strip makes a contribution:

$$\iint_{D_{t+dt}} F(x,y,t)\,dx\,dy - \iint_{D_t} F(x,y,t)\,dx\,dy \tag{92}$$

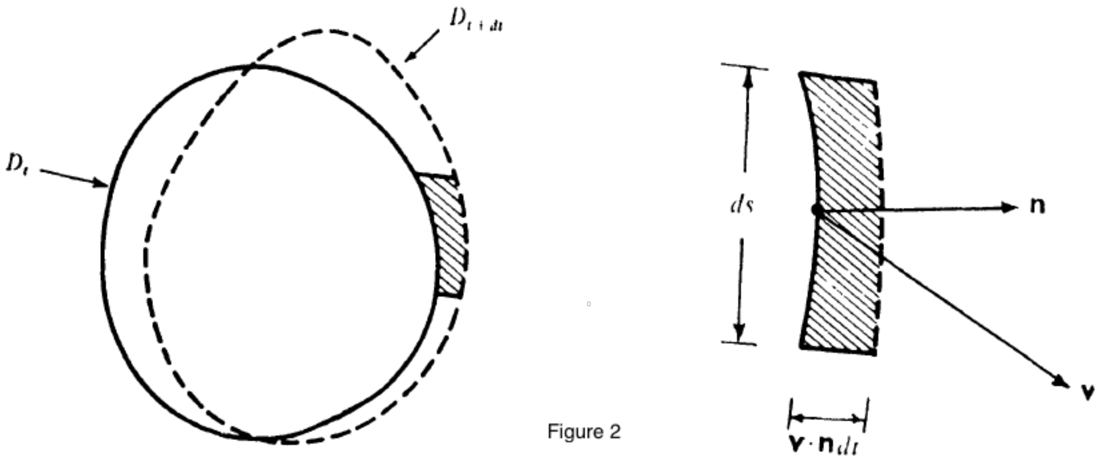


Figure 2

It can be seen from Figure 2 that the relevant contribution is given by:

$$F(x, y, t)(\mathbf{v}dt) \,.\, (\mathbf{n}\, ds) \tag{93}$$

In time dt the boundary which has velocity $\mathbf{v}$ moves a distance $\mathbf{v} \,.\, \mathbf{n}\, dt$ and the distance along the boundary is ds. Thus the product is an elemental area weighted by $F(x, y, t)$. Higher order terms are ignored and in a "mathematician's proof" you would use regularity of the function and region to show that the higher order terms don't matter. Thus Flanders arrives at this approximation:

$$\frac{1}{dt}\left(\iint_{D_{t+dt}} F(x, y, t)\, dx\, dy - \iint_{D_t} F(x, y, t)\, dx\, dy \right) \approx \int_{\partial D_t} F(x, y, t)\, \mathbf{v} \,.\, \mathbf{n}\, ds \tag{94}$$

where ∂ denotes the boundary as usual. To get $\mathbf{v} \,.\, \mathbf{n}\, ds$ Flanders rotates the unit tangent $(\frac{dx}{ds}, \frac{dy}{ds})$ backwards through a right angle to obtain $\mathbf{n} = (\frac{dy}{ds}, -\frac{dx}{ds})$. Thus we have:

$$\mathbf{v} \,.\, \mathbf{n}\, ds = (u, v) \,.\, (dy, -dx) = u\, dy - v\, dx \tag{95}$$

Thus:

$$\frac{d}{dt} \iint_{D_t} F(x, y, t)\, dx\, dy = \int_{\partial D_t} F(x, y, t)(u\, dy - v\, dx) \tag{96}$$

Equation (96) screams out for the application of Green's Theorem in the context of (91) and (96):

$$
\begin{aligned}
\frac{d}{dt}\iint_{D_t} F(x,y,t)\,dx\,dy &= \int_{\partial D_t} F(x,y,t)(u\,dy - v\,dx) + \iint_{D_t} \frac{\partial F}{\partial t}\,dx\,dy \\
&= \iint_{D_t} \left[\operatorname{div}(F\mathbf{v}) + \frac{\partial F}{\partial t}\right] dx\,dy
\end{aligned}
\tag{97}
$$

Recall that:

(1) $\iint_D \operatorname{div}\boldsymbol{\Phi}\,dA = \int_{\partial D} \boldsymbol{\Phi} \boldsymbol{.} \mathbf{n}\,ds$ and

(2) $\operatorname{div}(F\mathbf{v}) = \frac{\partial}{\partial x}(Fu) + \frac{\partial}{\partial y}(Fv) = \text{grad F} \boldsymbol{.} \mathbf{v} + F\operatorname{div}\mathbf{v}$.

Flanders then moves on to a 3-dimensional formula. He considers a fluid flowing through a region of space whose position is $\mathbf{x} = \mathbf{x}(\mathbf{u}, t)$ at time t of a particle of fluid originally at point $\mathbf{u}$. The velocity is $\mathbf{v} = \mathbf{v}(\mathbf{x}, t)$ at present time t of a particle now at position $\mathbf{x}$. Now the domain D_t moves with the flow and we suppose that we are given a function $F(\mathbf{x}, t)$ on the region of flow. Flanders then presents a mathematician's proof of a formula found in [8, pages 428-430] which was the subject of a "physicist's proof". The formula is:

$$
\begin{aligned}
\frac{d}{dt}\iiint_{D_t} F(\mathbf{x},t)\,dx\,dy\,dz &= \iint_{\partial D_t} F\mathbf{v} \boldsymbol{.} d\mathbf{S} + \iiint_{D_t} \frac{\partial F}{\partial t}\,dx\,dy\,dz \\
&= \iiint_{D_t} \left[\operatorname{div}(F\mathbf{v}) + \frac{\partial F}{\partial t}\right] dx\,dy\,dz
\end{aligned}
\tag{98}
$$

Here $d\mathbf{S}$ is an elemental vectorial area on the closed surface ∂D_t with:

$$
d\mathbf{S} = (dy\,dz, dz\,dx, dx\,dy) = \mathbf{n}\,dS \tag{99}
$$

where $\mathbf{n}$ is the outward unit normal and dS is the element of area.

The form of (99) may not be immediately obvious. Flanders was an expert in differential forms and the form of (99) reflects this background. Indeed, in [6.2, page 43] he defines the vectorial area element as $(\sigma_1\,\sigma_2)\,\mathbf{e_3}$ as a vector directed along the normal $\mathbf{e_3}$ with magnitude $\sigma_1\,\sigma_2$. This arises from the cross product of $d\mathbf{x} \times d\mathbf{x} = (\sigma_1\mathbf{e_1} + \sigma_2\mathbf{e_2}) \times (\sigma_1\mathbf{e_1} + \sigma_2\mathbf{e_2}) = 2(\sigma_1\sigma_2)\mathbf{e_3}$. You have to pay attention ot the ordering of terms in exterior algebra eg $\sigma_2\sigma_1(\mathbf{e_2}\times\mathbf{e_1}) = (-\sigma_1\sigma_2)(-\mathbf{e_1}\times\mathbf{e_2}) = (\sigma_1\sigma_2)\,\mathbf{e_3}$.

Another way of deriving (99) is to think of elementary Cartesian rectangles in each plane:

$$\begin{aligned}\mathbf{dS_x} &= dy\,dz\,\mathbf{n_x}\\ \mathbf{dS_y} &= dz\,dx\,\mathbf{n_y}\\ \mathbf{dS_z} &= dx\,dy\,\mathbf{n_z}\end{aligned} \tag{100}$$

The dS in (99) is interpreted to reflect the relevant factors in (100).

Flanders' "mathematician's" (as distinct from a physicist's) proof of (98) begins by defining the initial position as $\mathbf{u} = (u^1, u^2, u^3)$, the moving point is $\mathbf{x} = (x^1, x^2, x^3)$ and the velocity is $\dot{\mathbf{x}} = (v^1, v^2, v^3) = (\dot{x}^1, \dot{x}^2, \dot{x}^3)$ where the dot represents $\frac{\partial}{\partial t}$.

There is a domain C in **u**-space and for each t it evolves (which Flanders describes as an "imbedding $\Phi_t : C \twoheadrightarrow D_t$ of C into x - space. The mapping $(\mathbf{u}, t) \rightarrow \Phi_t(\mathbf{u})$ is assumed twice continuously differentiable and he writes $\Phi_t\mathbf{u}) = \mathbf{x}(\mathbf{u}, t)$. For each fixed t he writes the Jacobian matrix of Φ_t as:

$$\frac{\partial\mathbf{x}}{\partial\mathbf{u}} = \left[\frac{\partial x^i}{\partial u^j}\right] \tag{101}$$

The Jacobian matrix is non-singular everywhere and its inverse is $\frac{\partial\mathbf{u}}{\partial\mathbf{x}} = \left[\frac{\partial u^j}{\partial x^i}\right]$.

The determinant of the Jacobian matrix is $\left|\frac{\partial\mathbf{x}}{\partial\mathbf{u}}\right|$. Flanders next relies upon Jacobi's formula for the derivative of a determinant. Thus if $A = A(t)$ is a non-singular matrix function then:

$$\frac{|A|^{\cdot}}{|A|} = \text{trace}(\dot{A}A^{-1}) \tag{102}$$

(102) is proved in [5.2]

The next step is to apply (102) to the Jacobian matrix noting that, because of the assumed continuity of the derivatives:

$$\frac{\partial}{\partial t}\Big(\frac{\partial x^i}{\partial u^j}\Big) = \Big(\frac{\partial x^i}{\partial u^j}\Big)^{\cdot} = \frac{\partial \dot{x}^i}{\partial u^j} = \frac{\partial v^i}{\partial u^j} \tag{103}$$

Hence we have:

$$
\begin{aligned}
\text{trace}\Big\{\Big(\frac{\partial \mathbf{x}}{\partial \mathbf{u}}\Big)^{\boldsymbol{\cdot}}\Big(\frac{\partial \mathbf{x}}{\partial \mathbf{u}}\Big)^{-1}\Big\} &= \text{trace}\left\{\left[\frac{\partial v^i}{\partial u^j}\right]\left[\frac{\partial u^j}{\partial x^k}\right]\right\} \\
&= \sum_{i,j}\frac{\partial v^i}{\partial u^j}\,\frac{\partial u^j}{\partial x^i} \\
&= \sum_{i}\frac{\partial v^i}{\partial x^i} \\
&= \operatorname{div}\mathbf{v}
\end{aligned}
\tag{104}
$$

So using (102) we have:

$$
\frac{d}{dt}\left|\frac{\partial \mathbf{x}}{\partial \mathbf{u}}\right| = \left|\frac{\partial \mathbf{x}}{\partial \mathbf{u}}\right|(\ \operatorname{div}\mathbf{v}) \tag{105}
$$

Flanders now sets:

$$
f(t) = \iiint_{D_t} F(\mathbf{x},t)\,dx^1dx^2dx^3 \tag{106}
$$

Then the change of variables rule gives:

$$
f(t) = \iiint_{C} F[\mathbf{x}(\mathbf{u},t),t]\left|\frac{\partial \mathbf{x}}{\partial \mathbf{u}}\right| du^1du^2du^3 \tag{107}
$$

There is now a fixed domain and he differentiates:

$$
\begin{aligned}
\frac{df(t)}{dt} &= \iiint_C \left\{\Big[\sum_i \frac{\partial F}{\partial x^i}\,v^i + \frac{\partial F}{\partial t}\Big]\left|\frac{\partial \mathbf{x}}{\partial \mathbf{u}}\right| + F(\mathbf{x},t)\left|\frac{\partial \mathbf{x}}{\partial \mathbf{u}}\right|(\ \operatorname{div}\mathbf{v})\right\} du^1du^2du^3 \\
&= \iiint_{D_t}\left\{(\ \text{grad}\ F)\boldsymbol{.}\,\mathbf{v} + F\operatorname{div}\ \mathbf{v} + \frac{\partial F}{\partial t}\right\} dx^1dx^2dx^3 \\
&= \iiint_{D_t}\left\{\operatorname{div}\ (F\,\mathbf{v}) + \frac{\partial F}{\partial t}\right\} dx^1dx^2dx^3
\end{aligned}
\tag{108}
$$

since (grad F)$\centerdot$v+F div $\mathbf{v}$ = div ($F\,\mathbf{v}$). This mechanically demonstrated by:

$$(\text{ grad } F)\centerdot\mathbf{v} = \sum_i v^i\,\frac{\partial F}{\partial x^i} \tag{109}$$

$$F\,\text{div }\mathbf{v} = \sum_i F\,\frac{\partial v^i}{\partial x^i} \tag{110}$$

$$\text{div }(F\,\mathbf{v}) = \sum_i F\,\frac{\partial v^i}{\partial x^i} = \begin{pmatrix} \frac{\partial}{\partial x^1} \\ \frac{\partial}{\partial x^2} \\ \frac{\partial}{\partial x^3} \end{pmatrix} \centerdot \begin{pmatrix} Fv^1 \\ Fv^2 \\ Fv^3 \end{pmatrix} = (\text{ grad } F)\centerdot\mathbf{v} + F\,\text{div }\mathbf{v} \tag{111}$$

Thus after some work does (108) does equal (98).

8 Now for the physicist's proof

If we go to [8, pages 428-430] we can find a quite concise physicist type proof which runs like this. Sokolnikoff and Redheffer start with a volume integral:

$$\int_T u(P,t)\,d\tau \tag{112}$$

where $u(P,t)$ is a continuous scalar function in a simply connected region T. Because the position of points P in T can in principle vary with time, the region of integration will change with t. When the region is fixed so that T is independent of t and $u(P,t)$ and $\frac{\partial u}{\partial t}$ are continuous in T for all relevant t you get the standard Leibniz result:

$$\frac{d}{dt}\int_T u(P,t)\,d\tau = \int_T \frac{\partial u(P,t)}{\partial t}\,d\tau \tag{113}$$

But when the domain of integration varies with time the RHS of (113) should include another term to account for that change of region. To find this term consider the surface S bounding T at a certain time t. We suppose that S is such that the divergence theorem is applicable and consider an element ΔS of S with surface area $\Delta\sigma$. In a small time interval $(t, t+\Delta t)$ the points P of ΔS sweep out a region of space whose volume is:

$$\Delta\tau \approx (\mathbf{v} \cdot \mathbf{n})\, \Delta t\, \Delta\sigma \tag{114}$$

where $\mathbf{v}$ is the velocity of P and $\mathbf{n}$ is the exterior unit normal to the small surface element ΔS. Now we sum the products of $u(P,t)$ and the elemental areas over the whole surface S as follows:

$$\Delta I = \Delta t \sum_S u(P,t)\, (\mathbf{v} \cdot \mathbf{n})\, \Delta\sigma \tag{115}$$

Now we divide by Δt and pass to the limit and we get the formal result:

$$\frac{dI}{dt} = \int_S u(P,t)\ (\mathbf{v} \cdot \mathbf{n})\, d\sigma \tag{116}$$

This then is the term that should be added to the RHS of (113) due to the motion of the region. Thus:

$$\frac{d}{dt} \int_T u(P,t)\, d\tau = \int_T \frac{\partial u(P,t)}{\partial t}\, d\tau + \int_S u(P,t)\ (\mathbf{v} \cdot \mathbf{n})\, d\sigma \tag{117}$$

(There is a typo in equation 18-5 of [8], page 429 which is corrected in (117))

Clearly because u is a scalar function we have $u(\mathbf{v} \cdot \mathbf{n}) = (u\, \mathbf{v} \cdot \mathbf{n})$ and so we can write:

$$\int_S u(\mathbf{v} \cdot \mathbf{n})\, d\sigma = \int_T \text{div}\ (u\, \mathbf{v})\, d\tau \tag{118}$$

Thus we get:

$$\frac{d}{dt} \int_T u(P,t)\, d\tau = \int_T \Big[\frac{\partial u(P,t)}{\partial t} + \text{div}\ (u\, \mathbf{v})\Big]\, d\tau \tag{119}$$

This is what Flanders arrived in equation (98). As you can see it is a much shorter proof which actually contains the motivation for the more mathematical treatment of Flanders.

9 The style of proof one finds in PDE textbooks

In finishing this article I come finally to the style of proof one funds in PDE textbooks. Because these textbooks are usually directed at graduate level or senior undergraduate level they assume a degree of familiarity with harmonic theory. For instance in [7], a graduate PDE textbook, the author exhibits the following 5 line proof. Let u be a solution of the wave equation $u_{tt}(x,t) - \Delta u(x,t) = 0$ for $x \in \mathbb{R}^d$, $t > 0$.

The energy norm of u is defined as follows:

$$E(t) = \frac{1}{2}\int_{\mathbb{R}^d} \left\{ u_t(x,t)^2 + \sum_{i=1}^{d} u_{x^i}(x,t)^2 \right\} dx \tag{120}$$

It is claimed that:

$$\begin{aligned} \frac{dE}{dt} &= \int_{\mathbb{R}^d} \left\{ u_t u_{tt} + \sum_{i=1}^{d} u_{x^i}\, u_{x^i t} \right\} dx \\ &= \int_{\mathbb{R}^d} \left\{ u_t(u_{tt} - \Delta u) + \sum_{i=1}^{d} (u_t u_{x^i})_{x^i} \right\} dx \\ &=0 \end{aligned} \tag{121}$$

if "$u(x,t) = 0$ for sufficiently large $|x|$ " which may depend on t.

Clearly $\int_{\mathbb{R}^d} u_t(u_{tt} - \Delta u)\, dx = 0$ because u is a solution of the wave equation but why does $\int_{\mathbb{R}^d} \sum_{i=1}^{d} (u_t u_{x^i})_{x^i}\, dx = 0$? The author has used a fact about harmonic functions that if they are zero on a boundary of a volume then they are zero everywhere in that volume. Hence the derivatives in the sum are zero and the result follows.

10 References

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11 History

Created 25 November 2013
28 December 2015 - fixed some typos and added comment about resolution of

non-convergence of derived series - Churchill [3], pages 126-131
21 May 2017 - major extension with detailed proof of energy conservation in d dimensions. In the process there is a detailed examination of differentiation under the integral sign in more than one dimension (effectively how the Reynolds Transport Equation generalises Leibniz's rule).