Uniform continuity of sinc x

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1 Introduction

The function sinc x $= \frac{\sin x}{x}$ is well known to those who study Fourier theory. It is defined as follows:

$$\text{sinc x} = \frac{\sin x}{x} = \begin{cases} 1 & x = 0 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

The integral of sinc(x) over the whole real line or the half real line causes some calculus students big problems because it looks like it should be simple but it isn' t. Thus $\int_{-\infty}^{\infty} \text{sinc x}\, dx = \pi$ so that $\int_0^{\infty} \text{sinc x}\, dx = \frac{\pi}{2}$ due to the fact that sinc x is even. Don't even think about using integration by parts or a substitution. It takes complex integration theory or some relatively heavy duty classical analysis to do it properly. The oscillatory nature of $\sin x$ is responsible for the difficulties. Serious students of analysis or calculus will want to know the complete proof of why the integral converges but others may be happy with a more practical (ie "this works") understanding. If you are doing Fourier Theory you really do need to know how to derive the integral because it is such a fundamental part of the theory, although I won't present a proof here as I am more interested in the uniform continuity aspect of the function. The sinc function appears in lurid green flurorescent form on the front of Fry's electronics store in Sunnyvale, California. Students of Fourier Theory at Stanford can go and pay homage before it in their spare time! Here's the link: `http://www.frys-electronics-ads.com/cc/stores/view/8/Sunnyvale`

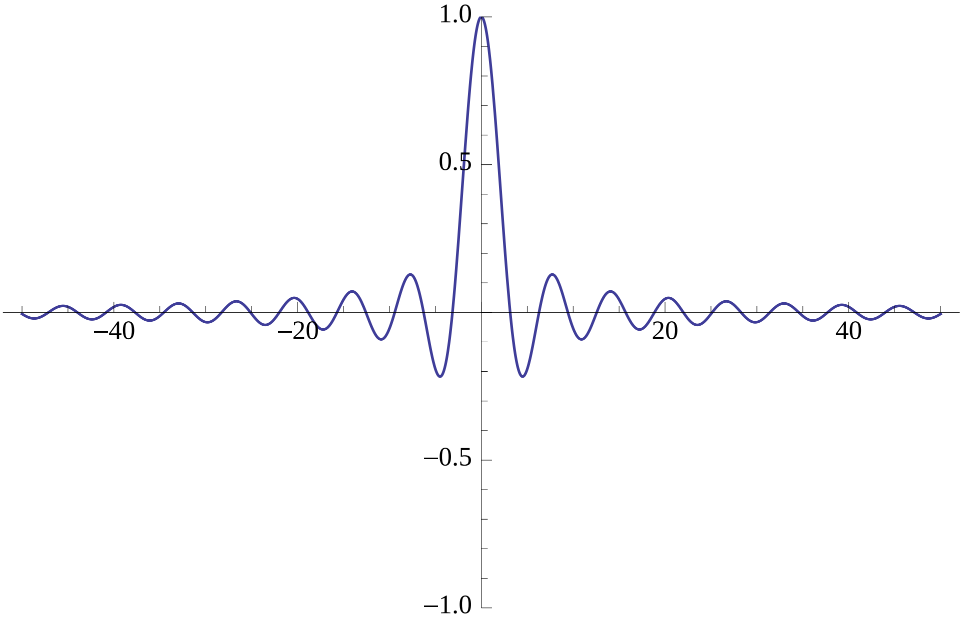
The sinc function appears in a wide variery of contexts including signal processing and probability theory. For instance, the uniform probability distribution on $(-a, a)$ has density $f(x) = \frac{1}{2a}$ and hence has characteristic function sinc(aξ). Recall that the characteristic function of f is defined as:

$$\psi(\xi) = \int_{-\infty}^{\infty} e^{i\xi x}\, f(x)\, dx \tag{2}$$

Hence in the case of the uniform distribution where $|x| \leq a$ we have:

$$\psi(\xi) = \int_{-a}^{a} \frac{e^{i\xi x}}{2a}\, f(x)\, dx = \frac{e^{i\xi x}}{2ai\xi}\Big]_{-a}^{a} = \frac{\cos(\xi a) + i\sin(\xi a) - [\cos(-\xi a) + i\sin(-\xi a)]}{2ai\xi}$$
$$= \frac{\sin(\xi a)}{\xi a} = \text{sinc}(\xi \text{a}) \tag{3}$$

The graph of the sinc function is set out below:



The uniform probability distribution is a species of the "box" function that is a fundamental part of Fourier Theory. The box function is also used in filtering signals.

Perhaps one of the most important applications of the sinc function and one which is of immense interest to electrical engineers is the role it plays in the Sampling Theorem in Fourier Theory. The Sampling Theorem can be expressed this way. If the Fourier Transform $F(\omega)$ of a signal function $f(t)$ is zero for all frequencies above $|\omega| \geq B$ (ie the signal is "band limited") then $f(t)$ can be uniquely determined from its sampled values $f(t) = \sum_{n=-\infty}^{\infty} f(nT)\ \text{sinc}\frac{(t-nT)}{T}$ where $T = \frac{1}{2B}$. This is quite a remarkable theorem and more information can be found here: `http://en.wikipedia.org/wiki/Nyquist-Shannon_sampling_theorem`

In quantum theory the sinc function arises in the analysis of a localized wave function of rectangular shape for a free particle [1]. The wave function is given by $\psi(x) = \int B_k \, e^{ikx} \, dk$ where the wave number k is taken to be a continuous variable. The coefficients are $B_k = \int \psi(x) \, e^{-ikx} \, dk$ by virtue of Fourier theory. If the rectangular function has width b then:

$$B_k = \int_{-\frac{b}{2}}^{\frac{b}{2}} e^{-ikx} \, dx = \frac{\sin(\frac{kb}{2})}{\frac{k}{2}} = b \, \mathrm{sinc}(\frac{kb}{2}) \tag{4}$$

The physical significance of this is that the wave function includes contributions of equal magnitude from positive and negative values of k. The consequence is that the average momentum of the particle is zero and the average position of the wave packet will not move right or left.

2 So how do you prove uniform continuity?

The only people who really care about this are people who specialise in analysis and Fourier theory in particular, but if you happen to inhabit those worlds even briefly you really do need to know how to prove uniform continuity unless you want to look like a complete dunce! The way the function is defined in (1) makes it clear that sinc x is continuous on the whole real line and hence it is continuous on any closed, bounded interval and so must be uniformly continuous on all such closed intervals no matter how large. So the function is indeed uniformly continuous on the entire real line. Note here that we cannot argue that because the product of two continuous functions is continuous (and hence replicate the same logic about uniform continuity on that basis) because $\frac{1}{x}$ blows up at $x = 0$. However, we know from calculus or analysis courses that $\lim_{x \to 0} \frac{\sin x}{x} = 1$ and so sinc x is continuous at $x = 0$ and it is obviously continuous everywhere else.

Now an $\epsilon - \delta$ proof is trickier as indeed they normally are for uniform continuity because it is a global rather than local property of a function. Recall that to prove that a function $f(x)$ is uniformly continuous, if you are given any $\epsilon > 0$ you must come up with a $\delta > 0$ such that for any x, y satisfying $0 < |x - y| < \delta$ you will have $|f(x) - f(y)| < \epsilon$.

First consider $1 \leq |x| < |y|$. We know by the Mean Value Theorem that there is an ξ between x and y such that:

$$|\frac{\sin x}{x} - \frac{\sin y}{y}| = |\Big(\frac{\sin \xi}{\xi}\Big)'| \, |x - y| = \Big|\frac{\xi \, \cos \xi - \sin \xi}{\xi^2}\Big| \, |x - y| \tag{5}$$

We see that:

$$\Big|\frac{\xi\,\cos\xi-\sin\xi}{\xi^2}\Big| \le \frac{|\xi\cos\xi|+|\sin\xi|}{\xi^2} \le \frac{|\xi|+1}{\xi^2} < 2 \tag{6}$$

since $|\xi|>1$

Thus for $\epsilon>0$ and $0<|x-y|<\frac{\epsilon}{2}$ we have:

$$|\frac{\sin x}{x}-\frac{\sin y}{y}| = \Big|\frac{\xi\,\cos\xi-\sin\xi}{\xi^2}\Big|\,|x-y| < 2|x-y| < 2\,\frac{\epsilon}{2} = \epsilon \tag{7}$$

ie our $\delta=\frac{\epsilon}{2}$

In the interval $(-1,1)$ we can use Taylor's theorem to approximate the derivative in the mean value expression by expanding about zero. The Taylor series expansions are as follows:

$$f(\xi)=\sin\xi=\xi-\frac{\xi^3}{3!}+R(\xi) \tag{8}$$

$$g(\xi)=\cos\xi=1-\frac{\xi^2}{2!}+R^{'}(\xi) \tag{9}$$

Recall that the general form of a Taylor series expansion about 0 looks like this:

$$f(x)=f(0)+xf^{'}(0)+\frac{x^2}{2!}f^{''}(0)+\cdots+\frac{x^n}{n!}f^{(n)}(0)+\frac{x^{n+1}}{(n+1)!}f^{(n+1)}(x^*) \tag{10}$$

where $x^*\in(0,x)$. The last term in (10) is the Lagrange form of the remainder and in the case of both $\sin x$ and $\cos x$ the absolute value of the derivative term in the remainder is bounded by 1 (since the n^{th} derivative is a multiple of $\sin x$ or $\cos x$). Thus:

$$|R(\xi)|\le\frac{|\xi|^4}{4!} \quad \text{and } |R'(\xi)|\le|\frac{|\xi|^3}{3!}| \tag{11}$$

Thus the derivative in (5) is bounded as follows:

$$\begin{aligned}\Big|\frac{\xi\,\cos\xi-\sin\xi}{\xi^2}\Big| &= \frac{|\,\xi(1-\frac{\xi^2}{2!}+R^{'}(\xi))-(\xi-\frac{\xi^3}{3!}+R(\xi))\,|}{|\xi|^2}\\ &= \frac{|-\frac{\xi^3}{3}+\xi\,R^{'}(\xi)-R(\xi)\,|}{|\xi|^2} \le \frac{\frac{|\xi^3|}{3}+|\xi|\,|R^{'}(\xi)|+|R(\xi)|}{|\xi|^2} \le \frac{\frac{|\xi^3|}{3}+|\xi|\,\frac{|\xi|^3}{6}+\frac{|\xi|^4}{24}}{|\xi|^2}\\ &= \frac{|\xi|}{3}+\frac{5|\xi|^2}{24} < 1 \text{ since } |\xi|<1\end{aligned} \tag{12}$$

Going back to (5) we see that if we choose $0 < |x-y| < \delta = \epsilon$ we will have:

$$|\frac{\sin x}{x} - \frac{\sin y}{y}| = \left|\frac{\xi\,\cos\xi - \sin\xi}{\xi^2}\right| |x-y| < |x-y| < \epsilon \tag{13}$$

So if we choose as our "global" $\delta = \frac{\epsilon}{2}$, whenever $0 < |x-y| < \delta$ it will follow that $|\frac{\sin x}{x} - \frac{\sin y}{y}| < \epsilon$. In this proof we needed to get different estimates for the two tailed intervals and the middle "hump" interval. This is not uncommon as the following demonstration shows.

Note that the interval you choose in relation to a proof of uniform continuity makes a difference to the type of estimate you need to make. For instance, if we were only interested in the interval $(0,1)$, we could use the fact that on this interval $\sin\xi \leq \xi$ (in fact this holds for $0 \leq \xi \leq \pi/2$) so that the derivative $\frac{\xi\,\cos\xi - \sin\xi}{\xi^2} \geq \frac{\xi\cos\xi - \xi}{\xi^2} \geq \frac{\cos\xi - 1}{\xi} \geq \cos\xi - 1 \geq -1$. Thus the absolute value of the derivative is in the interval $(0,1]$ so that $|\frac{\sin x}{x} - \frac{\sin y}{y}| = \left|\frac{\xi\,\cos\xi - \sin\xi}{\xi^2}\right| |x-y| \leq |x-y| < \epsilon$. In this case we needn't use Taylor's theorem.

3 Proving that if f is Riemann integrable on every interval $[a,b]$ and $\int_{-\infty}^{\infty} |f(t)|\,dt$ converges, then the Fourier transform $\hat{f}$ is uniformly continuous

We take $\epsilon > 0$ as given. Now because $\int_{-\infty}^{\infty} |f(t)|\,dt$ converges, we can make the tails of the integral arbitrarily small by going far enough out, just like with a convergent infinite series. This means that there is a $R(\epsilon) > 0$ (obviously how far we need to go out will depend on how small we make ϵ) such that $\int_{|t| \geq R(\epsilon)} |f(t)|\,dt < \frac{\epsilon}{4}$. That deals with the tails.

To make an estimate of the middle "rump" we observe that because f is Riemann integrable on $[-R(\epsilon), R(\epsilon)]$ it must be bounded on that interval (it will be continuous due to the Fundamental Theorem of Calculus) so we can say that $\sup_{t\in[-R(\epsilon),R(\epsilon)]} |f(t)| = B(\epsilon)$. Now to show that the Fourier transform $\hat{f}$ converges uniformly we have to show that is some δ such that whenever $0 < |\xi - \nu| < \delta$ we have $|\hat{f}(\xi) - \hat{f}(\nu)| < \epsilon$ and to do this we break the relevant integral into the tails and the rump as follows, recalling that the Fourier transform of f is $\hat{f}(\xi) = \int_{-\infty}^{\infty} f(t)\, e^{-2\pi i \xi t}\,dt$:

$$
\begin{aligned}
|\hat{f}(\xi)-\hat{f}(\nu)| &= \Big| \int_{-\infty}^{\infty} f(t)\,(e^{-2\pi i\xi t}-e^{-2\pi i\nu t})\,dt \Big| \\
&\le \Big| \int_{|t|\ge R(\epsilon)} f(t)\,(e^{-2\pi i\xi t}-e^{-2\pi i\nu t})\,dt \Big| + \Big| \int_{-R(\epsilon)}^{R(\epsilon)} f(t)\,(e^{-2\pi i\xi t}-e^{-2\pi i\nu t})\,dt \Big| \\
&\le \int_{|t|\ge R(\epsilon)} \Big| f(t)\,(e^{-2\pi i\xi t}-e^{-2\pi i\nu t})\Big|\,dt + \int_{-R(\epsilon)}^{R(\epsilon)} \Big| f(t)\,(e^{-2\pi i\xi t}-e^{-2\pi i\nu t})\Big|\,dt \\
&\le 2\int_{|t|\ge R(\epsilon)} |f(t)|\,dt + 2R(\epsilon) \sup_{t\in[-R(\epsilon),R(\epsilon)]} |f(t)(e^{-2\pi i\xi t}-e^{-2\pi i\nu t})| \\
&\le 2\frac{\epsilon}{4} + 2R(\epsilon)\,B(\epsilon) \sup_{t\in[-R(\epsilon),R(\epsilon)]} |(e^{-2\pi i\xi t}-e^{-2\pi i\nu t})| \\
&\le \frac{\epsilon}{2} + 2R(\epsilon)\,B(\epsilon) \sup_{t\in[-R(\epsilon),R(\epsilon)]} |\xi t-\nu t| \le \frac{\epsilon}{2} + 4\pi\,R(\epsilon)^3\,B(\epsilon)\,|\xi-\nu| \qquad (14)
\end{aligned}
$$

Now if we choose $|\xi-\nu| \le \frac{\epsilon}{8\pi\,R(\epsilon)^3\,B(\epsilon)+1}$ then the last inequality will become:

$$
\frac{\epsilon}{2}+4\pi\,R(\epsilon)^3\,B(\epsilon)\,|\xi-\nu| = \frac{\epsilon+8\pi\,R(\epsilon)^3\,B(\epsilon),|\xi-\nu|}{2} < \frac{\epsilon+8\pi\,R(\epsilon)^3\,B(\epsilon)\,\frac{\epsilon}{8\pi\,R(\epsilon)^3\,B(\epsilon)+1}}{2} < \frac{2\epsilon}{2} = \epsilon \qquad (15)
$$

This establishes the uniform continuity of the Fourier transform.

Note that the Mean Value theorem is used in (14) for if we let $g(t) = e^{-2\pi it}$ so that there exists a λ between ξ and ν (all of which are in $[-R(\epsilon), R(\epsilon)]$) such that:

$$
|g(\xi t)-g(\nu t)| = |e^{-2\pi\xi it}-e^{-2\pi\nu it}| = |g^{'}(\lambda t)|\,|\xi t-\nu t| = |-2\pi i\lambda\,e^{-2\pi i\lambda t}|\,|t|\,|\xi-\nu| \le 2\pi|\lambda|\,|t|\,|\xi-\nu| \qquad (16)
$$

Hence $\sup_{t\in[-R(\epsilon),R(\epsilon)]} 2\pi|\lambda|\,|t|\,|\xi-\nu| \le 2\pi\,R(\epsilon)^2\,|\xi-\nu|$ because λ is also in $[-R(\epsilon), R(\epsilon)]$.

If we define $f(x) = 1$ for $x \in [-1,1]$ and $f(x) = 0$ otherwise then f clearly satisfies the condition of absolute integrability since $\int_{-\infty}^{\infty}|f(t)|\,dt = \int_{-1}^{1} dt = 2$. However, $\int_{-\infty}^{\infty}|\hat{f}(\xi)|\,d\xi$ does not converge. So what is $\int_{-\infty}^{\infty}|\hat{f}(\xi)|\,d\xi$? It is none other than the absolute integral of the sinc function. Thus:

$$
\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)\,e^{-2\pi i\xi x}\,dx = \int_{-1}^{1} e^{-2\pi i\xi x}\,dx = \frac{-e^{-2\pi i\xi x}}{2\pi i\xi}\Bigg]_{-1}^{1} = \frac{2\sin(2\pi\xi)}{2\pi\xi} \qquad (17)
$$

To demonstrate that $\int_{-\infty}^{\infty}|\hat{f}(\xi)|\,d\xi$ does not converge we make the following estimates:

$$\int_{-(n+1)}^{n+1} |\hat{f}(\xi)|\, d\xi \geq \sum_{k=0}^{n} \int_{k+\frac{1}{8}}^{k+\frac{3}{8}} |\hat{f}(\xi)|\, d\xi \geq \sum_{k=0}^{n} \int_{k+\frac{1}{8}}^{k+\frac{3}{8}} \frac{2}{\sqrt{2}\,\xi}\, d\xi > \sum_{k=0}^{n} \frac{2}{\sqrt{2}} \frac{\frac{1}{4}}{(k+\frac{3}{8})}$$
$$= \sum_{k=0}^{n} \frac{1}{2\sqrt{2}} \frac{1}{(k+\frac{3}{8})} > \sum_{k=0}^{n} \frac{1}{2\sqrt{2}} \frac{1}{(k+1)} \to \infty \text{ as } n \to \infty \quad (18)$$

In making these estimates a couple of points are worth making:

1. First, we ignore the negative tail in the integral which certainly ensures that the integral of $\hat{f}(\xi)$ will be at least as big as the series of positive tail terms.

2. Second, we choose the intervals of integration so that you get $\sin(2\pi(k+\frac{1}{8})) = \sin(2\pi k + \frac{\pi}{4})$ and $\sin(2\pi(k+\frac{3}{8}) = \sin(2\pi k + \frac{3\pi}{4}))$ etc. The reason for doing this is that for ξ in the interval $\Big[\sin(2\pi k+\frac{\pi}{4})), \sin(2\pi k+\frac{3\pi}{4}))\Big]$ the value of $\sin(2\pi\xi) \geq \sin(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$.

3. Third, the term $\frac{1}{\xi}$ in the integrand in each sub-interval dominates the rectangular box formed by using the right-hand ξ-value, eg, the first box has area $\frac{1}{4} \times \frac{1}{(k+\frac{3}{8})} > \frac{1}{4} \times \frac{1}{(k+1)}$. All subsequent boxes have the same width of $\frac{1}{4}$.

4 Concluding remarks

Although we have established by laborious $\epsilon - \delta$ techniques that sinc x is uniformly continuous, if we had known some Fourier theory we could have ridden on the coat tails of a general theorem about the uniform continuty of Fourier transforms to get to the same result. The oscillatory nature of the sin function is sufficient in combination with $\frac{1}{x}$ to ensure that the integral $\int_{-\infty}^{\infty} \text{sinc x}\, dx$ converges but the decay of $\frac{1}{x}$ is not sufficiently fast to ensure that the integral of the absolute value of the function converges.

5 References

[1] A P French and Edwin F Taylor , ”An Introduction ot Quantum Physics”, Van Nostrand, 1987, pages 331- 336

[2] T W Körner, ”Fourier Analysis”, Cambridge University Press, 1988.

6 Appendix

Note that for $x \in [-\pi/2, \pi/2]$, $|\sin x| \leq |x|$. The following diagram which deals with $\sin x$ on the interval $[0, \frac{\pi}{2}]$ demonstrates the inequality.