Using the heat equation to motivate the idea of the Fourier transform

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1 Introduction

In a previous paper ([1]) I sought to give some intuition behind the Fourier and Laplace transforms. This paper seeks to add a further dimension to the understanding of the Fourier transform by taking the heat equation as a specific vehicle for illuminating some of the concepts. In what follows I use the term 'Fourier transform ' to encompass what are in fact Fourier integrals which are nothing more than a species of trigonometric integrals since the Fourier transform $\int_{-\infty}^{\infty} f(x)e^{-2\pi i x\xi}\,dx$ in its exponential form is simply the sum of sine and cosine integrals.

2 The heat equation problem - discrete eigenvalues

For simplicity we start with this basic set up. Our temperature in space and time is $u(x,t)$ and it satisfies the following conditions:

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} \tag{1}$$

with boundary conditions:

$$\frac{\partial u(0,t)}{\partial x} = 0 \tag{2}$$

$$\frac{\partial u(1,t)}{\partial x} = 0 \tag{3}$$

This problem gives rise to **discrete** eigenvalues and it can be shown (see Appendix 1 for all the gory details) that a sequence of functions $u_n(x,t) = e^{-n^2\pi^2 t}\cos n\pi x$ for $n = 0, 1, 2, 3, \ldots$ satisfy (1)-(3).

This needs to be quickly checked:

$$\frac{\partial u(x,t)}{\partial t} = -n^2\pi^2 e^{-n^2\pi^2 t} \cos n\pi x \tag{4}$$

$$\begin{aligned} \frac{\partial u(x,t)}{\partial x} &= -\, n\pi e^{-n^2\pi^2 t} \sin n\pi x \\ \frac{\partial^2 u(x,t)}{\partial x^2} &= -\, n^2\pi^2 e^{-n^2\pi^2 t} \cos n\pi x \end{aligned} \tag{5}$$

Checking the boundary conditions:

$$\frac{\partial u(0,t)}{\partial x} = -n\pi e^{-n^2\pi^2 t} \sin 0 = 0 \tag{6}$$

$$\frac{\partial u(1,t)}{\partial x} = -n\pi e^{-n^2\pi^2 t} \sin n\pi = 0 \tag{7}$$

Now there are theorems that allow us to say that any linear combination of the u_n is a solution to the problem:

$$u(x,t) = \sum_{n=0}^{\infty} c_n e^{-n^2\pi^2 t} \cos n\pi x \tag{8}$$

where the c_n are constants.

Now let us just assume for the moment that we have a solution to the heat problem which is a function of a **continuous** parameter ν. This is mathematically promiscuous but nevertheless productive as we will see. Thus our basic solution looks like this:

$$u(x,t) = e^{-\nu^2 t} \cos \nu x \tag{9}$$

where the constant is assumed to be 1.

If we now integrate with respect to the parameter ν between $-\infty$ and ∞ our infinite linear superposition solution (8) looks like this for $t > 0$:

$$u(x,t) = \int_{-\infty}^{\infty} e^{-\nu^2 t} \cos \nu x \, d\nu \tag{10}$$

Now if you can differentiate under the integral (and you can in this case - see the Appendix in [2]) you can see that (10) is indeed a solution:

$$\frac{\partial u(x,t)}{\partial x} = \int_{-\infty}^{\infty} -v \, e^{-\nu^2 t} \sin \nu x \, d\nu \tag{11}$$

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \int_{-\infty}^{\infty} -v^2\, e^{-\nu^2 t} \cos \nu x \, d\nu \tag{12}$$

and this equals:

$$\frac{\partial u(x,t)}{\partial t} = \int_{-\infty}^{\infty} -v^2\, e^{-\nu^2 t} \cos \nu x \, d\nu \tag{13}$$

Thus (9) satisfies (1).

We need to check that (10) satisfies the boundary conditions:

$$\frac{\partial u(0,t)}{\partial x} = \int_{-\infty}^{\infty} -v\, e^{-\nu^2 t} \sin 0 \, d\nu = 0 \tag{14}$$

and this boundary condition for $t > 0$:

$$\frac{\partial u(1,t)}{\partial x} = \int_{-\infty}^{\infty} -v\, e^{-\nu^2 t} \sin \nu \, d\nu = 0 \tag{15}$$

because $\int_{-\infty}^{\infty} -v\, e^{-\nu^2 t} \sin \nu \, d\nu = -\frac{1}{2t}\sqrt{\frac{\pi}{t}}\, e^{-\frac{1}{4t}}$ and the discussion around (27).

Thus the artifice of moving from an infinite linear superposition to an integral transform seems to work. It in fact allows us to show the following:

$$u(x,t) = \int_{-\infty}^{\infty} e^{-\nu^2 t} \cos \nu x \, d\nu = \sqrt{\frac{\pi}{t}}\, e^{-\frac{x^2}{4t}} \tag{16}$$

To do this integration let:

$$\nu^2 t = \lambda^2 \text{ and } \alpha = \frac{x}{\sqrt{t}} \tag{17}$$

then:

$$\cos \nu x = \cos \lambda\alpha \text{ and } d\nu = \frac{d\lambda}{\sqrt{t}} \tag{18}$$

Hence:

$$\int_{-\infty}^{\infty} e^{-\nu^2 t} \cos \nu x \, d\nu = \frac{1}{\sqrt{t}} J(\alpha) \tag{19}$$

where:

$$J(\alpha) = \int_{-\infty}^{\infty} e^{-\lambda^2} \cos \lambda\alpha \, d\lambda \tag{20}$$

We can differentiate $J(\alpha)$ with respect to α under the integral sign (the integrand is well behaved enough for this) to set up a differential equation that we will solve to get $J(\alpha)$. Thus we have:

$$J^{'}(\alpha) = -\int_{-\infty}^{\infty} \lambda e^{-\lambda^2} \sin \lambda\alpha \, d\lambda \tag{21}$$

Equation (21) screams for integration by parts as follows:

$$\begin{aligned} J^{'}(\alpha) =& -\int_{-\infty}^{\infty} \lambda e^{-\lambda^2} \sin \lambda\alpha \, d\lambda \\ =& \int_{-\infty}^{\infty} \sin \lambda\alpha \, d(\frac{1}{2}e^{-\lambda^2}) \\ =& \Big[\underbrace{\frac{1}{2}e^{-\lambda^2} \sin \lambda\alpha \Big|_{-\infty}^{\infty}}_{=0} - \frac{1}{2}\int_{-\infty}^{\infty} \alpha e^{-\lambda^2} \cos \lambda\alpha \, d\lambda \Big] \\ =& -\frac{\alpha}{2} J(\alpha) \end{aligned} \tag{22}$$

We can solve (22):

$$J(\alpha) = Ae^{-\frac{\alpha^2}{4}} \tag{23}$$

for some constant A. But

$$J(0) = \int_{-\infty}^{\infty} e^{-\lambda^2} \, d\lambda = \sqrt{\pi} \implies A = \sqrt{\pi} \tag{24}$$

Hence we have our solution from (19) using (17):

$$\begin{aligned} u(x,t) =& \frac{1}{\sqrt{t}} J(\alpha) \\ =& \sqrt{\frac{\pi}{t}} \, e^{-\frac{\alpha^2}{4}} \\ =& \sqrt{\frac{\pi}{t}} \, e^{-\frac{x^2}{4t}} \text{ for } t > 0 \end{aligned} \tag{25}$$

Checking this solution with the boundary conditions (2)-(3) we see that:

$$\frac{\partial u(x,t)}{\partial x} = -\frac{x}{2t}\sqrt{\frac{\pi}{t}} \, e^{-\frac{x^2}{4t}} \tag{26}$$

so $\frac{\partial u(0,t)}{\partial x} = 0$ while:

$$\frac{\partial u(1,t)}{\partial x} = -\frac{1}{2t}\sqrt{\frac{\pi}{t}} \, e^{-\frac{1}{4t}} \tag{27}$$

which holds for all $t > 0$, and it can be seen that when t is near zero the RHS of (27) is very small ie approaches 0. When t is large the RHS behaves like $\frac{-\sqrt{\pi}}{2t^{\frac{3}{2}}}$ and this goes to 0. For small t the behaviour can be seen more formally by letting $t = \frac{1}{n}$ and letting $n \to \infty$ so that $t \to 0^+$. In that case the RHS of (27) becomes $\frac{-\sqrt{\pi} n^{\frac{3}{2}}}{2} e^{-\frac{n}{4}}$ and this rapidly approaches 0 because the exponential dominates $n^{\frac{3}{2}}$. For instance when $t = 0.001$ the RHS of (27) is $-7.48039 \times 10^{-105}$. So condition (3) is satisfied for all $t > 0$.

Standing back from this we took a discrete eigenvalue problem which gave us an infinite linear superposition of solutions and we essentially treated that as a continuous integral in quite a cavalier fashion without any of the approximation arguments referred to in [1]. But it worked!

3 The heat equation problem - continuous eigenvalues

We now turn to a problem which involves **continuous** eigenvalues. The set up is as follows:

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} \text{ for } x > 0, t > 0 \tag{28}$$

with boundary conditions:

$$u(x,t) = 0 \text{ for } t > 0 \tag{29}$$

$$|u(x,t)| < M \text{ for } x > 0, t > 0 \tag{30}$$

where $M > 0$ is a constant, and

$$u(x,0) = f(x) \text{ for } x > 0 \tag{31}$$

Replicating the separation of variables approach we are looking for $U(x,t) = X(x)T(t)$ and to achieve the boundedness in (30) we require that $X(x)$ and $T(t)$ be bounded to ensure that linear combinations of their product are bounded. This gives rise to the following:

$$X^{''}(x) + \lambda X(x) = 0 \text{ for } x > 0 \tag{32}$$

$$X(0) = 0 \text{ , and } |X(x)| < M_1 \tag{33}$$

$$T^{'}(t) + \lambda T(t) = 0 \text{ for } t > 0 \tag{34}$$

$$|T(t)| < M_2 \tag{35}$$

where M_1, M_2 are constants.

Solving (32) subject to (33) as usual (assuming $\lambda \neq 0$ so we get non-trivial solutions) we get:

$$X(x) = a\cos\sqrt{\lambda}\,x + b\sin\sqrt{\lambda}\,x \tag{36}$$

for constants a, b.

Now $X(0) = 0 \implies a = 0$ so :

$$X(x) = \sin\sqrt{\lambda}\,x \tag{37}$$

where $b = 1$ for convenience. Now if $\sqrt{\lambda}$ is complex, ie $\sqrt{\lambda} = c + di$ where $c, d \in \mathbb{R}$, we will not get a bounded solution for positive x. To see this we express $\sin\sqrt{\lambda}\,x$ in exponential form as follows (noting that $|e^{i\theta}| = 1$ for all θ):

$$\begin{aligned}
|\sin\sqrt{\lambda}\,x| &= \left|\frac{e^{i(c+di)x} - e^{-i(c+di)x}}{2i}\right| \\
&= \frac{|e^{icx}e^{-dx} - e^{-icx}e^{dx}|}{2} \\
&\geq \frac{\left||e^{icx}e^{-dx}| - |e^{-icx}e^{dx}|\right|}{2} \\
&= \frac{\left||e^{icx}|\,|e^{-dx}| - |e^{-icx}|\,|e^{dx}|\right|}{2} \\
&= \frac{\left||e^{-dx}| - |e^{dx}|\right|}{2} \\
&= \frac{\left|e^{-dx} - e^{dx}\right|}{2} \quad \text{since the exponentials are both} > 0
\end{aligned} \tag{38}$$

Irrespective of whether d is positive or negative, $\frac{\left|e^{-dx}-e^{dx}\right|}{2}$ grows exponentially and thus $\sin\sqrt{\lambda}\,x$ is unbounded.

From this it follows that $\sqrt{\lambda} = \alpha$ must be real and positive (if it is negative, because $\sin -\alpha x = -\sin\alpha x$ the sign just gets picked up in a constant of superposition). Thus $\sin\alpha x$ is bounded.

Thus we have that the eignevalues $\lambda = \alpha^2$ are continuous and $\sin\alpha x$ are the eigenfunctions.

The corresponding functions for $T(t)$ are (see Appendix 1):

$$T(t) = e^{-\alpha^2 t} \tag{39}$$

Clearly (39) is bounded.

Now we invoke our promiscuous technique for using an integral over the continuous positive parameter α to get a generalised linear superposition of the products $X(x)T(t)$ which will be our solution:

$$u(x,t) = \int_0^\infty g(\alpha) e^{-\alpha^2 t} \sin \alpha x \, d\alpha \tag{40}$$

which must satisfy all the boundary conditions so that we must have the following from (31);

$$f(x) = u(x,0) = \int_0^\infty g(\alpha) \sin \alpha x \, d\alpha \text{ for } x > 0 \tag{41}$$

Equation (41) will be a Fourier sine integral representation of the function f if we have:

$$g(\alpha) = \frac{2}{\pi} \int_0^\infty f(\xi) \sin \alpha\xi \, d\xi \text{ for } \alpha > 0 \tag{42}$$

(see [3], pages 113-119). For more details see Appendix 2.

Our solution to the problem therefore looks like this, at least formally:

$$u(x,t) = \frac{2}{\pi} \int_0^\infty e^{-\alpha^2 t} \sin \alpha x \int_0^\infty f(\xi) \sin \alpha\xi \, d\xi \, d\alpha \tag{43}$$

To solve (43) we note that:

$$2 \sin \alpha x \sin \alpha\xi = \cos[\alpha(x-\xi)] - \cos[\alpha(x+\xi)] \tag{44}$$

so (43) becomes the following after we exchange the order of integration (this is justified by Fubini's Theorem which certainly applies to the functions at issue in the integrals we are looking at. For a proof of Fubini's Theorem in the general Lebesgue context see [4], 79-89):

$$u(x,t) = \frac{1}{\pi} \int_0^\infty f(\xi) \int_0^\infty \left[e^{-\alpha^2 t} \Big(\cos[\alpha(x-\xi)] - \cos[\alpha(x+\xi)] \Big) \right] d\alpha \, d\xi \tag{45}$$

To perform the inner integral in (45) we use (16) but note that since the integrand in that case is even we need to halve the answer, ie:

$$\int_{-\infty}^\infty e^{-\nu^2 t} \cos \nu x \, d\nu = \sqrt{\frac{\pi}{t}} \, e^{-\frac{x^2}{4t}} \tag{46}$$

implies that:

$$\int_0^\infty e^{-\nu^2 t} \cos \nu x \, d\nu = \frac{1}{2}\sqrt{\frac{\pi}{t}}\, e^{-\frac{x^2}{4t}} \tag{47}$$

Hence (45) becomes:

$$u(x,t) = \frac{1}{2\sqrt{\pi t}} \int_0^\infty f(\xi) \left[e^{-\frac{(x-\xi)^2}{4t}} - e^{-\frac{(x+\xi)^2}{4t}} \right] d\xi \text{ for } t > 0 \tag{48}$$

4 Conclusions

As can be seen from the development of the solutions to the heat equation in the discrete and continuous eigenvalue cases, the use of a Fourier integral or transform actually arises from the process of solving the differential equations (which were Sturm-Liouville type problems). This use of integration was already well understood in the 18^{th} century in a practical sense even if the theory lacks the insights of this century. The motivation of the Fourier transform was and is the solution of differential equations and that is its basic motivation.

5 Appendix 1

To solve the system set out in (1)-(3) we use the method of separation of variables where:

$$U(x,t) = X(x)T(t) \tag{49}$$

This leads to:

$$\frac{\partial U(x,t)}{\partial x} = X^{'}(x)T(t) \tag{50}$$

$$\frac{\partial^2 U(x,t)}{\partial x^2} = X^{''}(x)T(t) \tag{51}$$

and

$$\frac{\partial U(x,t)}{\partial t} = X(x)T^{'}(t) \tag{52}$$

Thus:

$$\begin{aligned} X^{''}(x)T(t) =& X(x)T^{'}(t) \\ \text{ie } \frac{X^{''}(x)}{X(x)} =& \frac{T^{'}(t)}{T(t)} = -\lambda \end{aligned} \tag{53}$$

The last line of (53) is based on the fact that the left hand side is a function of x alone and so cannot vary with t while the right hand side is a function of t alone and can't vary with x. Their common value is a constant $-\lambda$ which we have to investigate.

From:

$$\frac{\partial U(0,t)}{\partial x} = 0 \tag{54}$$

we have that:

$$X^{'}(0)T(t) = 0 \tag{55}$$

which holds for all $t > 0$. If we want something other than the trivial solution $U(x,t) \equiv 0$ then we have:

$$X^{'}(0) = 0 \tag{56}$$

Similarly, from:

$$\frac{\partial U(1,t)}{\partial x} = 0 \tag{57}$$

we have that:

$$X^{'}(1)T(t) = 0 \tag{58}$$

and so have:

$$X^{'}(1) = 0 \tag{59}$$

We now need to solve:

$$X^{''}(x) + \lambda X(x) = 0 \tag{60}$$

subject to (56) and (59).

A general solution is:

$$X(x) = a\cos\sqrt{\lambda}\,x + b\sin\sqrt{\lambda}\,x \tag{61}$$

Since:

$$X^{'}(x) = -a\sqrt{\lambda}\sin\sqrt{\lambda}\,x + b\sqrt{\lambda}\cos\sqrt{\lambda}\,x \tag{62}$$

Hence $X^{'}(0) = 0$ implies that $b\lambda = 0$ and since we are assuming $\lambda \neq 0$ we must have that $b = 0$ and so (taking $a = 1$ for convenience without any loss of generality in what follows):

$$X(x) = \cos\sqrt{\lambda}\,x \tag{63}$$

Note that of we allowed $\lambda = 0$ then we would get linear solutions for $X(x)$ and $T(t)$ and that ultimately leads to a trivial solution for $U(x,t)$ taking into account the boundary conditions (convince yourself by going through the detail).

From (59)and (63) we have that:

$$X^{'}(1) = -\sqrt{\lambda}\,\sin\sqrt{\lambda} = 0 \implies \sqrt{\lambda} = n\pi \text{ for } n = 1, 2, \ldots \tag{64}$$

Thus our discrete eigenvalues are:

$$\lambda = n^2\pi^2 \text{ for } n = 1, 2, \ldots \tag{65}$$

and so

$$X(x) = \cos n\pi\, x \tag{66}$$

From (53) we see that the general solution to:

$$\frac{T^{'}(t)}{T(t)} = -\lambda \tag{67}$$

is:

$$T(t) = ce^{-\lambda t} = ce^{-n^2\pi^2 t} \tag{68}$$

for some constant c which, for convenience, we take as 1. Thus our solution to the problem (1)-(3) is:

$$u_n(x, t) = e^{-n^2\pi^2 t}\cos n\pi x \text{ for } n = 1, 2, \ldots \tag{69}$$

as advertised earlier.

6 Appendix 2

Under suitable assumptions we which don't need to worry about for the purposes of this paper (but see [3] sec. 51 for details) $f(x)$ can be expressed by the Fourier integral formula as:

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(\xi)\cos[\alpha(\xi - x)]\,d\xi d\alpha \tag{70}$$

for $-\infty < x < \infty$.

(70) can also be written as:

$$f(x) = \int_0^{\infty}[A(\alpha)\cos\alpha x + B(\alpha)\sin\alpha x]\,d\alpha \tag{71}$$

where:

$$\begin{aligned} A(\alpha) =& \frac{1}{\pi}\int_{-\infty}^{\infty} f(\xi)\cos\alpha\xi\, d\xi \\ B(\alpha) =& \frac{1}{\pi}\int_{-\infty}^{\infty} f(\xi)\sin\alpha\xi\, d\xi \end{aligned} \tag{72}$$

If $f(x)$ is an odd function then $f(\xi)\cos\alpha\xi$ and $f(\xi)\sin\alpha\xi$ are odd and even functions of ξ respectively. Hence we will have:

$$\begin{aligned} A(\alpha) =& \frac{1}{\pi}\int_{-\infty}^{\infty} f(\xi)\cos\alpha\xi\, d\xi = 0 \\ B(\alpha) =& \frac{1}{\pi}\int_{-\infty}^{\infty} f(\xi)\sin\alpha\xi\, d\xi = \frac{2}{\pi}\int_{0}^{\infty} f(\xi)\sin\alpha\xi\, d\xi \end{aligned} \tag{73}$$

and so we get the Fourier sine integral formula:

$$f(x) = \frac{2}{\pi}\int_0^{\infty} \sin\alpha x \int_0^{\infty} f(\xi)\sin\alpha\xi\, d\xi\, d\alpha \tag{74}$$

The analogous Fourier cosine integral formula is:

$$f(x) = \frac{2}{\pi}\int_0^{\infty} \cos\alpha x \int_0^{\infty} f(\xi)\cos\alpha\xi\, d\xi\, d\alpha \tag{75}$$

If $f(x)$ is only defined for $x > 0$ we can make it odd, for instance, by defining it to be equal to $-f(-x)$ when $x < 0$ so that $f(-x) = -f(x)$ for all x.

7 References

[1] Peter Haggstrom, The intuition behind the Fourier and Laplace transforms, `https://gotohaggstrom.com/The%20intuition%20behind%20the%20Fourier%20and%20Laplace%20transforms.pdf`

[2] Peter Haggstrom, Basic Fourier integrals, `https://gotohaggstrom.com/Basic%20Fourier%20integrals.pdf`

[3] Ruel V Churchill, , Fourier Series and Boundary Value Problems, Second Edition, McGraw-Hill, 1963.

[4] Elias M Stein and Rami Shakarchi, Real Analysis: Measure theory, Integration and Hilbert spaces, Princeton Lectures in Analysis III, Princeton University Press, 2005.

8 History

Created 26 April 2020 15 May 2020 - 3 pathetic typos corrected