1

What do schmucks* and the arc sine law have in common?

by Peter Haggstrom, March 2011, www.gotohaggstrom.com

1.1. Background to the arc sine law

The arc sine law is really only well known to serious students of probability theory, mathematical statistics and actuarial science, yet it provides fundamental insights for investors who are bombarded with marketing material about investment performance. It has remarkably simple beginnings which can be encapsulated by the following example given by the famous William Feller in his textbook "Introduction to Probability Theory and Its Applications" [1]:

"Suppose that a great many coin-tossing games are conducted simultaneously at the rate of one per second, day and night, for a whole year. On the average, in one out of ten games the last equalization will occur before 9 days have passed, and the lead will not change during the following 356 days. In one out of twenty cases the last equalization takes place within 2 1/4 days, and in one out of a hundred cases it occurs within the first 2 hours and 10 minutes"

Feller makes the following point about sign changes:

".. one should naively expect that in a prolonged coin tossing game the observed number of changes of lead should increase roughly in proportion to the duration of the game. In a game that lasts twice as long, Peter should lead about twice as often. This intuitive reasoning is false. We shall show that, in a sense to be made precise, the number of sign changes of lead in n trials increases only as $\sqrt{n}$: in 100n trials one should expect only 10 times as many changes of lead as in n trials. This proves once more that the waiting times between successive equalizations are likely to be fantastically long".[2]

The arc sine law underpins the analytical fact that in long coin tossing games one of the players remains on the winning side for practically the whole time with the other player on the losing side. This conflicts with so-called "intuition" which is in fact incredibly unreliable. "Intuition" is unreliable for the simple reason that the logical foundation for the fundamental results in this area rely upon some technical combinatorial results which hide the structural form of the results. I cannot perform the complex calculations required in my head and although there may be some talented individuals in existence who can make intelligent estimates, the vast majority of people either cannot perform the basic calculations at all or, if they can, would have to use a computer.

What this means is that when all you have is faulty "intuition" to inform your investment decision making, you will inevitably misperceive the risks you are actually facing. I have tested some of the coin tossing probability examples in Feller's book on a room of about 300 financial planners and not one of them got anywhere near the correct probabilities. These are the people who advise investors on what they should do and yet they are completely in the dark about the risks involved in even the simplest of experiments. This is not surprising given that they are merely product sales people, but even trained people would still misconstrue the actual probabilities. Indeed, Feller found precisely that when he got highly trained people to comment on some coin tossing data which most thought was "extreme" when it wasn't extreme at all. However, the fundamental point remains - if people misperceive the probabilities in simple coin tossing experiments, how on earth are they going to correctly judge the probabilities in much more complex real world phenomena. The answer is, of course, that they won't. In the investment context they will be pushed in various directions by clever marketing from asset managers and financial advisers. They will then be surprised when the world does not perform as they were led to believe.

Against this background an understanding of the arc sine law is relevant in an investment context simply because life companies and asset managers devise products which reflect pricing of investment patterns which, though more complex than a simple coin tossing game, harbour the same sorts of counter-intuitive behaviour. Intuition is almost certainly wrong in accurately predicting what the risks are.

For instance, a number of Australian asset managers who also operate life companies offer so-called longevity retirement products to the public. These are high fee products which pander to emotions of greed (getting a slice of the action when markets go up) and fear (protecting capital when markets go down). The mug punter is easily manipulated by slick marketing which masks the nature of the bet they are making with the life company which guarantees them some rate of income (inevitably a low rate) if their money runs out because they have outlived their life expectancy.

What they do is this. They provide the punter with a statistic that is quite correct - eg that a 65 year old Australian male has about a 50% chance of living to 93, that is, more than their life expectancy of 18.54 years. They push the punter into high cost funds (which can even be over-priced index funds) which fall into the "balanced" category which means they gave a relatively low probability of generating a negative return. For instance the typical Australian balanced fund has a 13.23% chance of having a negative year. The product providers don't tell you that nor do they tell you the "barrier" probabilities of falls of x% or rises of y% from certain levels. The punter's investment balance has to completely evaporate before the life company guarantee kicks in and the products are designed to discourage or limit the amount that can be drawn down as a pension. In essence, they seek to force you to draw down relatively modest amounts which are unlikely to cause the investment account to disappear and hence trigger the guarantee. They thus take inflated fees for many years for an event that has an extremely low probability of occurring. The law does not require them to provide the punter with the relevant probabilities and meaningful comparisons. All the pricing is completely obscured inside a "black box".

Applying some basic binomial theory to this type of situation it is easily seen that if the probability of a negative year is 0.1323 then the probability of a positive year is 0.8677 and hence the probability that in 28 years there will be more positive years than negative years is:

$$\sum_{j=15}^{28} \binom{28}{j} 0.8677^{j}\ 0.1323^{28-j} = 0.999997 \quad \text{where } \binom{n}{j} = \frac{n!}{(n-j)!\, j!} \tag{1.1}$$

While this does not prove that the investment balance will not evaporate (because we have no information about the scale of negative positive returns in this simple example), it provides a high level "smell test" indication of the basic direction of the bet the punter is making with the life company. You can be sure that the actuaries have done much more sophisticated calculations which would be designed to give them confidence that the life company will be on the winning side of the bet. Casino operators are in the business of winning.

The other dimension to these products is how the life company's fund performs and whether other policy holders in the same fund have a superior call on the assets. When markets are down generally, as occurred during the global financial crisis, the assets of the life company fund will probably be down. Life companies run statutory funds which house the assets to pay life policies and when you read the relevant contracts closely you find that it is possible for these pre-existing policy owners to be preferred if there is a general deficiency in the fund. Thus it is in principle possible that the "guarantee" may never eventuate at all or the amount provided is insufficient to meet the amount you thought you would get. Of course, if the insurer does something really stupid like the American giant insurer AIG, it can teeter on insolvency. The venerable old British insurance firm Equitable LLife nearly became insolvent in 1999 because it could not pay policyholders what it contracted to pay them so it took them to court to force them to accept policy bonus cuts. Equitable Life lost in the High Court and could not pay the 1.5 billion pounds due to policy holders and when it tried to find a buyer for the business none was forthcoming so the insurer closed to new business. A glorious triumph for the actuarial and investment professions! So if you think you are an "informed" investor the chances are you aren't and what the alleged professionals are up to in the bowels of the organisation may ultimately have quite different outcomes for your investment than you expected.

* The investment world is full of schmucks, which is a wonderfully pejorative Yiddish word for a gold plated fool. The investment intelligentsia is genetically predisposed to preying on schmucks and there is no shortage of material to practice upon in order to refine one's skills. The internet is full of forums venting the opinions of people who could not even understand equation (1.1) yet they live in a delusional world in which they believe that they can better the professional guys who are rigging the roulette wheel.

1.2. The formal statement of the arc sine law

The probability that up to an including time 2n (Feller uses the word "epoch") the last visit to the origin occurs at time 2n is given by:

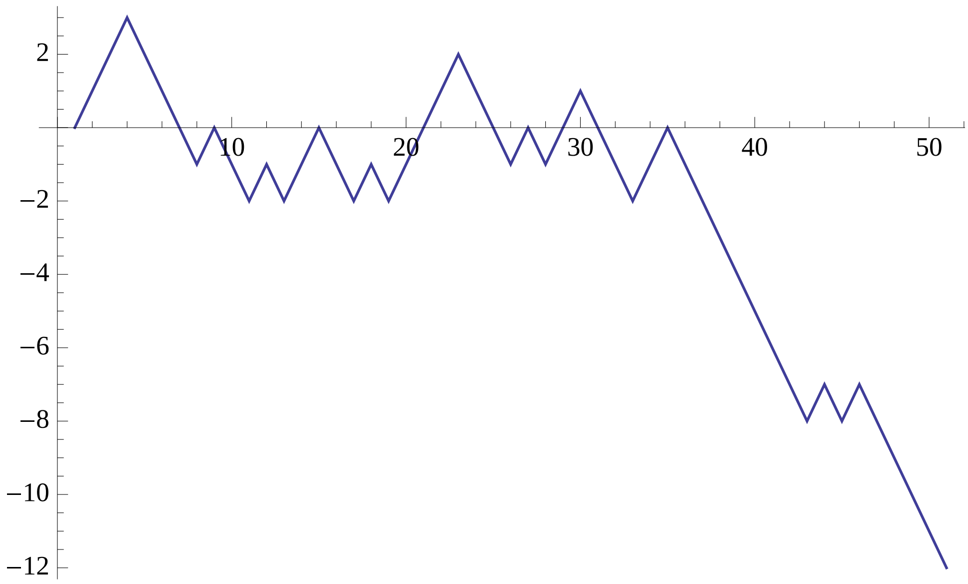
$$\alpha_{2k,2n} = u_{2k}\, u_{2n-2k} \quad \text{where } k = 0, 1, ..., n \qquad (1.2)$$

The symbols and their meaning have to be explained for this to make any sense. This is done below.

The first point to make is that the way Feller derives the arc sine law is essentially combinatorial and this was quite mysterious at the time ie the 1950s. Mathematicians of the stature of Paul Levy had found arc sine law behavior in the context of Brownian motion while Paul Erdos and Mark Kac had derived an arc sine law limit for the number of positive partial sums in a sequence of mutually independent random variables. E Sparre Andersen shocked people in 1953-54 when he showed that many facets of the fluctuation theory of sums of independent random variables are of an essentially combinatorial character. Kai Lai Chung and Feller proved a variant of (1.2) by much more complex methods. For citations on this history see [3]

In what follows I flesh out many of the calculations in Feller's book since he merely states many results and leaves it to the reader to fill in the gaps. I have summarised some of his proofs which tend to be pretty clear once you have established all the relevant building blocks.

The starting point for (1.2) is a formula for counting paths for by p plus ones and q minus ones such that n = p + q. If one thinks of δ_t as being a +1 or -1 occurring at time t, the arrangement $(\delta_1, \delta_2, ..., \delta_n)$ represents a polygonal line whose k^{th} vertex has ordinate $s_k = \sum_{j=1}^{k} \delta_j$. In other words you have a path.



The *Mathematica* code for such a path is given below.

```
randomWalk[n_] := Module[ {steps, walk},
                    steps = Table[ 2 RandomInteger[] - 1, {n}];
                    walk = FoldList[Plus, 0, steps];
                    ListPlot[walk, PlotJoined -> True] ];
randomWalk[50]
```

It is common to call the "length" of the path n and there are 2^n paths of length n which are assumed to have equal probability ie each path has probability 2^{-n}. If p of the δ_t are positive and q are negative then n = p + q. If the path is to go through the point (n,x) where $s_n = \sum_{j=1}^{n} \delta_j$ = x then we must have x = p - q.

$$\text{A path running from the origin to } (n, x) \text{ necessarily involves } n \text{ and } x \text{ of the form } n = p + q \text{ and } x = p - q \tag{1.3}$$

It is a standard combinatorial technique to then conclude that the number of paths from the origin to (n,x) is simply the number of ways of choosing p positive δ_t from the n = p + q total steps. Thus the number of different ways is $N_{n,\,x}$ where :

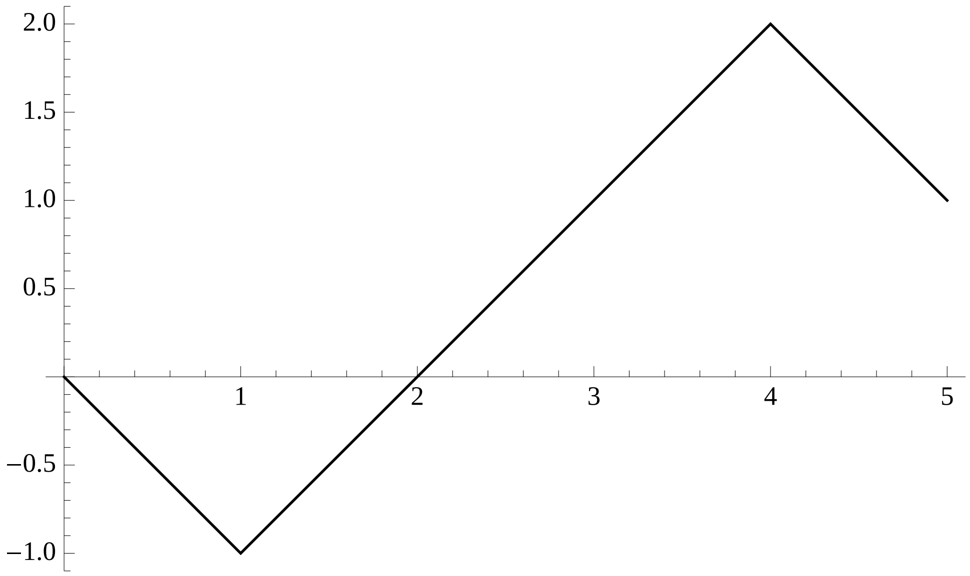
$$N_{n,x} = \binom{p+q}{p} = \binom{p+q}{q} \tag{1.4}$$

A more detailed proof is given in Problem 38 (Item 6). Formula (1.4) only holds n and x are of the form in (1.3) so the convention is that $N_{n,\,x}$ = 0 in all other cases.

To develop a general theory one needs to consider the individual steps as random variables $X_k$and the positions of the "particle"(which could be anything really) being S_k where:

$$S_n = \sum_{k=1}^{n} X_k \text{ and } S_0 = 0 \tag{1.5}$$

For example, the following graph represents the points X_1= -1, X_2= +1, X_3 = +1, X_4 = + 1, X_5 = -1 and S_1 = -1, S_2= 0, S_3= 1, S_4= 2, S_5= 1



The probability that the particle is at point r at time n is $p_{n,r}$= prob {S_n= r}

Now we know from (1.4) that the number of paths from the origin to (n,r) is $N_{n,\,r}$ and so the probability (under the assumption made that all the 2^npaths are equally likely) that the particle visits r is:

$$p_{n,r} = \text{prob}\,\{S_n = r\} = \frac{\binom{n}{\frac{n+r}{2}}}{2^n} \quad \text{Note that from (1.4) } n = \qquad (1.6)$$

$$p + q \text{ and } r = p - q \text{ so that } n + r = 2\,p \text{ and so } p = \frac{n+r}{2}$$

Because the number of paths is always integral you have to interpret $\binom{n}{\frac{n+r}{2}}$ as zero unless $\frac{n+r}{2}$ is an integer between 0 and n.

There is an important concept of "return" to the origin at time k ie where S_k= 0. Clearly k must be even (2υ) since if you go up υ steps you must go down υ steps to get back to the origin. The probability of a return to the origin is denoted by $u_{2\upsilon}$ and from (1.6) it follows that:

$$u_{2\upsilon} = \frac{\binom{2\upsilon}{\upsilon}}{2^{2\upsilon}} \qquad (1.7)$$

One can approximate (1.7) using Stirling's formula which is:

$$n! \sim \sqrt{2\pi}\, n^{n+\frac{1}{2}}\, e^{-n} \tag{1.8}$$

Hence $\frac{\binom{2\upsilon}{\upsilon}}{2^{2\upsilon}} = \frac{(2\upsilon)!}{\upsilon!\,\upsilon!} 2^{-2\upsilon}$

$$\sim \frac{\sqrt{2\pi}\,(2\upsilon)^{2\upsilon+\frac{1}{2}}\, e^{-2\upsilon}\, 2^{-2\upsilon}}{\sqrt{2\pi}\,\upsilon^{\upsilon+\frac{1}{2}}\, e^{-\upsilon}\,\sqrt{2\pi}\,\upsilon^{\upsilon+\frac{1}{2}}\, e^{-\upsilon}}$$

$$= \frac{2^{2\upsilon+\frac{1}{2}}\,\upsilon^{2\upsilon}\,\upsilon^{\frac{1}{2}}\, 2^{-2\upsilon}}{\sqrt{2\pi}\,\upsilon^{2\upsilon}\,\upsilon}$$

$$= \frac{1}{\sqrt{\pi\upsilon}}$$

So, $u_{2\upsilon} \sim \frac{1}{\sqrt{\pi\upsilon}}$ (1.9)

In other words the ratio of $\frac{\binom{2\upsilon}{\upsilon}}{2^{2\upsilon}}$ and $\frac{1}{\sqrt{\pi\upsilon}}$ approaches 1 for large n

The next concept is that of first return to the origin and this will occur at time 2υ if the following conditions hold:

$$S_k \neq 0 \text{ for } k = 1 \text{ to } 2\upsilon - 1 \text{ but } S_{2\upsilon} = 0 \tag{1.10}$$

The probability for first return is denoted by $f_{2\upsilon}$ where $f_0 = 0$

There is a relationship between $u_{2\upsilon}$ and $f_{2\upsilon}$ which developed as follows. Either:

(1) there is a first return to the origin at time 2n or;

(2) the first return occurs at a time 2k < 2n (note that as discussed above these times must be even) which is followed by a return 2n - 2k time units later.

This is the standard way to approach a recurrence probability relation with random phenomena. You develop the sum of mutually exclusive event probabilities.

Now there are $2^{2k} f_{2k}$ paths ending with a first return to the origin and there are $2^{2n-2k} u_{2n-2k}$ paths from the point (2k,0) to (2n,0). Hence the probability of (2) is:

$$\frac{2^{2k}\, f_{2k}\, 2^{2n-2k}\, u_{2n-2k}}{2^{2n}} = f_{2k}\, u_{2n-2k} \tag{1.11}$$

For n ≥ 1 (1.11) represents a set of mutually exclusive probabilities hence:

$$u_{2n} = f_2 u_{2n-2} + f_4 u_{2n-4} + \dots + f_{2n} u_0 \qquad (1.12)$$

In words, the probability of a return to the origin at time 2n is the iterated sum of probabilities of a first return at time 2 times the probability of a return 2n - 2 units later etc.

In problems Feller asks readers to prove the following things- see [4]:

$$u_{2n} = (-1)^n \binom{\frac{-1}{2}}{n} \qquad (1.13)$$

$$f_{2k} = (-1)^{n-1} \binom{\frac{1}{2}}{n} \qquad (1.14)$$

Using (1.13) and (1.14) is then possible to prove (1.12) but you need to know the following hypergeometric identity:

For any positive integers $a,\ b,\ n$: $\binom{a}{0}\binom{b}{n} + \binom{a}{1}\binom{b}{n-1} + \dots + \binom{a}{n}\binom{b}{0} = \binom{a+b}{n}$ This is established in the Appendix below. (1.15)

To prove (1.13) and (1.14) you first need to note the following:

$$\binom{m}{n} = \frac{m(m-1)(m-2)\dots(m-n+1)(m-n)!}{(m-n)!\,n!} = \frac{m(m-1)(m-2)\dots(m-n+1)}{n!} \qquad (1.16)$$

$$u_{2n} = \frac{\binom{2n}{n}}{2^{2n}} = \frac{2n(2n-1)(2n-2)(2n-3)(2n-4)\dots\ 4.3\ .2\ .1}{2^{2n}\, n!\, n!}$$

$$= \frac{2^n n!\,(2n-1)(2n-3)(2n-5)\dots 3.1}{2^{2n}\, n!\, n!}$$ because there are n terms of the form 2(n -k) on the top and n terms of the form (n -k) (ie the components of n!) on the bottom.

Hence $$u_{2n} = \frac{(2n-1)(2n-3)(2n-5)\dots 3.1}{2^n\, n!} \qquad (1.17)$$

Now $(-1)^n \binom{\frac{-1}{2}}{n} = (-1)^n \dfrac{\frac{-1}{2}\left(\frac{-1}{2}-1\right)\left(\frac{-1}{2}-2\right)\left(\frac{-1}{2}-3\right)\dots\left(\frac{-1}{2}-n+1\right)}{n!}$ using (1.16)

$= (-1)^n \dfrac{(-1)^n\, 1\times 3\times 5\dots(2n-1)}{2^n\, n!}$ noting that there are n terms on the top of the form $\frac{-(-1-2k)}{2}$

$$= \quad \frac{1\times 3\times 5...(2n-1)}{2^n\, n!} \quad \text{since } (-1)^{2n} = 1$$

$$= \quad u_{2n}$$

To prove that $f_{2k} = (-1)^{n-1}\binom{\frac{1}{2}}{n}$ we need a preliminary result. Feller proves the following remarkable result (see [5]):

The probability that no return to the origin occurs up to and including time 2n is the same as the probability that a return occurs at time 2n ie

$$P\{S_1 \neq 0, \ ..., S_{2n} \neq 0\} = P\{S_{2n} = 0\} = u_{2n} \tag{1.18}$$

Feller then argues from this that "saying that a first return occurs at epoch 2n amounts to saying that the conditions $S_1 \neq 0$, ..., $S_{2k} \neq 0$ are satisfied for k = n -1 but not for k =n"

Using (1.18) this means that:

$$f_{2n} = u_{2n-2} - u_{2n} \tag{1.19}$$

To see this you have to note that for two events A and B, P{A - B} = P{A} - P{AB} where A - B = A\B = $A\bar{B}$ ie the set comprising elements of A but not B. Now A can be written as the union of two mutually exclusive sets thus: A = (A - B) $\cup$ AB hence P{A} = P{ A - B} + P{AB}.

In the case at hand we have A = { $S_1 \neq 0$, ..., $S_{2k} \neq 0$ for k = n -1} and B = {$S_{2k} = 0$} and using the definition of f_{2n} and (1.18) one gets (1.19).

Now the preliminary result we need to show is that:

$$f_{2n} = \frac{1}{2n-1} u_{2n} \tag{1.20}$$

This follows from (1.19) using the fact that $u_{2n} = \frac{\binom{2n}{n}}{2^{2n}}$. The details are straightforward but here they are for completeness:

$$f_{2n} = u_{2n-2} - u_{2n}$$

$$= \frac{\binom{2n-2}{n-1}}{2^{2n-2}} - \frac{\binom{2n}{n}}{2^{2n}}$$

$$= \frac{(2n-2)!}{(n-1)!\,(n-1)!\,2^{2n-2}} - \frac{(2n)!}{n!\,n!\,2^{2n}}$$

$$= \frac{(2n-2)!}{(n-1)!\,(n-1)!\,2^{2n-2}}\left[1 - \frac{(2n)(2n-1)}{4n^2}\right]$$

$$= \frac{(2n-2)!}{(n-1)!\,(n-1)!\,2^{2n-2}}\left[\frac{4n^2 - 4n^2 + 2n}{4n^2}\right]$$

$$= \frac{(2n)(2n-2)!}{n!\,n!\,2^{2n}}$$

$$= \frac{1}{2n-1}\,\frac{(2n)!}{n!\,n!\,2^{2n}}$$

$$= \frac{1}{2n-1}\,u_{2n}$$

It follows from (1.20) that:

$$\sum_{k=i}^{\infty} f_{2k} = 1 \qquad (1.21)$$

The implication of (1.21) is that in a long coin tossing experiment it indeed becomes practically certain that there will be an equalisation, thus "validating" intuition, however, the number of trials required to approach certainty is not consistent with intuition.

To prove (1.21) one could observe that it is certain that there will be a first return at some stage. Alternatively a more analytic proof runs as follows:

Now $|\sum_{k=1}^{n} f_{2k} - 1| = |(\sum_{k=1}^{n} u_{2k-2} - u_{2k}) - 1| = |(u_0 - u_2) + (u_2 - u_4) + ... + (u_{2n-4} - u_{2n-2})$ $+ (u_{2n-2} - u_{2n}) - 1|$

$= | u_0 - u_{2n} - 1 |$

$= | 1 - u_{2n} - 1 |$ since $u_0 = 1$

$= u_{2n}$

As noted above in (1.9) u_{2n} is like $\frac{1}{\sqrt{\pi n}}$ for large n so that $|\sum_{k=1}^{n} f_{2k} - 1|$ can be made arbitrarily small by taking n large enough. Thus $\lim_{n\to\infty} \sum_{k=1}^{n} f_{2k} = 1$

The probability that no equalisation occurs in 100 tosses of a coin is thus $u_{100} = \frac{\binom{100}{50}}{2^{100}} = 0.0795$

or about 8%. If you had an 8% of winning the lottery you would think you had pretty good chances of winning so this level of probability for no equalization seems quite high and would not be predicted by most people.

If 2n coin tossing trials are conducted where n is large one could investigate the last trial at which the number of heads equals tails. This number will necessarily be even, 2k ($0 \le k \le n$) and "intuition" might suggest that there would be frequent changes of lead so that k would be close to n. However, the arc sine law analytically demonstrates that k is symmetrically distributed in the sense that any k has the same probability as n - k. According to Feller (see [6]) this rather amazing fact was found "empirically by computer and verified theoretically without knowledge of the exact distribution" (1.2). The symmetry implies that $k > \frac{n}{2}$ and $k < \frac{n}{2}$ are equally likely so that there is a probability of 50% that no equalization occurs in the game irrespective of the length of the game. Moreover, the boundary values (ie the extremes) give the highest probabilities, thereby conflicting with "intuition".

Feller's proof of the arc sine law

Feller's original proof was much more complicated than the version in the Third Edition of his textbook. The arc sine law is that up to and including time 2n, the last visit to the origin occurs at 2k is given by (1.2). Thus the paths must satisfy $S_{2k} = 0$ and $S_{2k+1} \neq 0, \ldots, S_{2n} \neq 0$ for k = 0, 1, 2, ..., n. The first 2k vertices can be chosen in $2^{2k} u_{2k}$ different ways and using (1.18) there are $2^{2n-2k} u_{2n-2k}$ ways of choosing the next (2n - 2k) vertices running from (2k,0) which can be treated as a new origin for the purposes of the analysis.

Thus the required probability is $\frac{2^{2k} u_{2k} 2^{2n-2k} u_{2n-2k}}{2^{2n}} = u_{2k} u_{2n-2k}$

Using the estimate derived in (1.9) we have that for large n:

$$u_{2k}\, u_{2n-2k} \sim \frac{1}{\sqrt{\pi k}} \frac{1}{\sqrt{\pi(n-k)}} = \frac{1}{\pi \sqrt{k(n-k)}} = \frac{1}{\pi n \sqrt{\frac{k}{n}\left(1 - \frac{k}{n}\right)}} \qquad (1.22)$$

Now let $x_k = \frac{k}{n}$ and $f(x) = \frac{1}{\pi \sqrt{x(1-x)}}$ in the context of (1.22) and it is seen that

$\alpha_{2k,\,2n} = u_{2k}\, u_{2n-2k} \sim \frac{1}{n} f(x_k)$

The approximation $\frac{1}{n} f(x_k)$ suggests that integration may be a goer. Accordingly we try the following integral to get some further insights:

$$\sum_{k<nx} \alpha_{2k,2n} \sim \sum_{k<nx} \frac{1}{n} f(x_k) \sim \qquad (1.23)$$

$$\int_0^x \frac{1}{\pi\sqrt{u(1-u)}}\,du \text{ for fixed } x \text{ in } (0,\ 1) \text{ and } n \text{ large enough}$$

Now $\int_0^x \frac{1}{\pi\sqrt{u(1-u)}}\,du$ can be integrated as follows:

$$\int_0^x \frac{1}{\pi\sqrt{u(1-u)}}\,du = \int_0^x \frac{1}{\pi\sqrt{u}\sqrt{1-\left(\frac{u}{\sqrt{u}}\right)^2}}\,du$$

$$= \int_0^x \frac{1}{\pi\sqrt{u}\sqrt{1-\left(\sqrt{u}\right)^2}}\,du$$

Now let $v = \sqrt{u}$ so that $dv = \frac{1}{2\sqrt{u}}du$

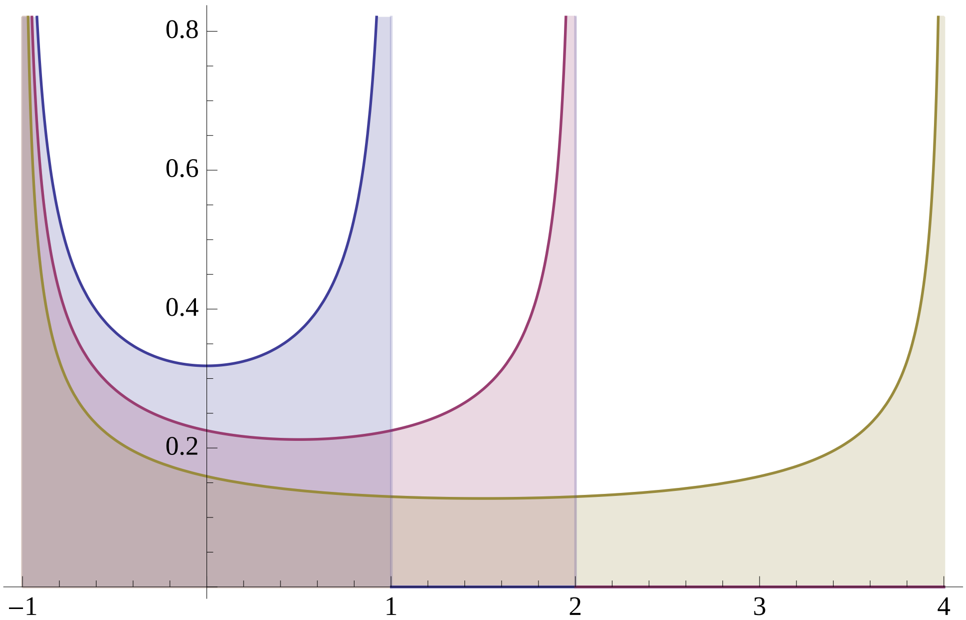
Hence $\int_0^x \frac{1}{\pi\sqrt{u}\sqrt{1-\left(\sqrt{u}\right)^2}}\,du = \int_0^{\sqrt{x}} \frac{2}{\pi\sqrt{1-v^2}}\,dv$

$= \frac{2}{\pi}\sin^{-1}\sqrt{x}$

Hence $\sum_{k<nx} \alpha_{2k,2n}$ behaves like $\frac{2}{\pi}$ (1.24)

$\sin^{-1}\sqrt{x}$ and the reference to "arc sine law" is justified.

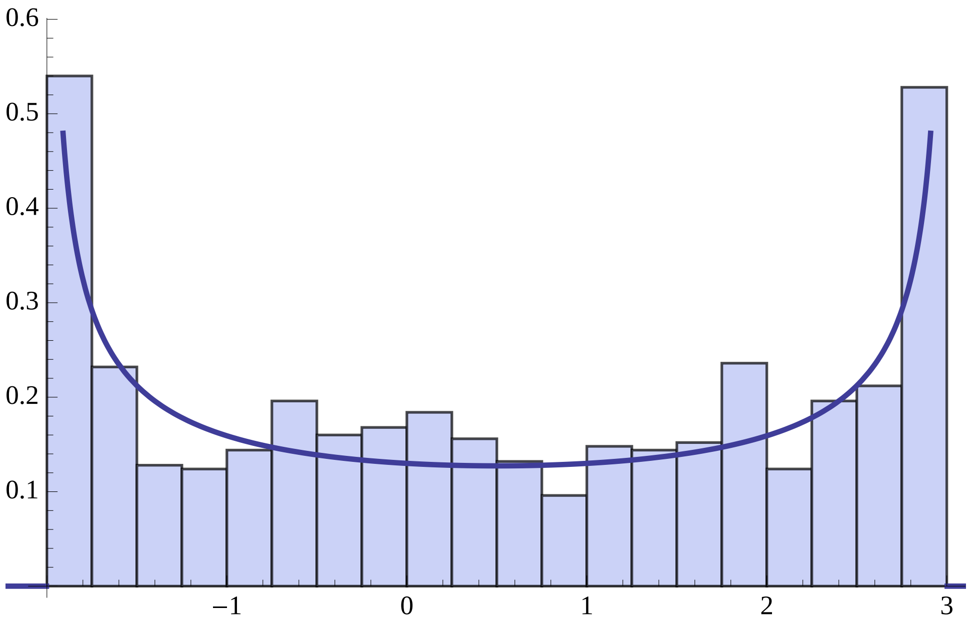
The arc sine graph has the following form:



The *Mathematica* code to generate the graphs is:

```
Plot[Evaluate@Table[PDF[ArcSinDistribution[{-1,max}],x],{max,{1,2,4}}],{x,-1,4},Filling→
Axis]
```

The following graph shows the arc sine function in the context of a density histogram of a sample:



The *Mathematica* code used to generate the above graph is:

```
sample=RandomVariate[ArcSinDistribution[{-2,3}],10^3;
edist=EstimatedDistribution[sample, ArcSinDistribution[{min,max}]];
Show[Histogram[sample,20, "PDF"],Plot[PDF[edist,x],{x,-3,4},PlotStyle→Thick]]
```

There are other results that Feller notes (see [7]).

The probability that in the time interval from 0 to $2n$ a particle spends $2k$ time units on the positive side and $2n - 2k$ time units on the negative side equals $\alpha_{2k, 2n}$ (1.25)

If $0 < x < 1$,
the probability that $\leq$ nx time units are spent on the positive side and $\geq$ $(1 - x)n$ on the negative side tends to $\frac{2}{\pi} \sin^{-1}\sqrt{x}$ as $n \to \infty$ (1.26)

The probability that in 20 tossings of a coin each player leads 10 times is $\alpha_{10,10} = u_{10}\, u_{10} =$ 0.06056 which would conflict with "intuition". It gets worse. If n is large there is a probability of 0.20 that a particle will spend about 97.6% of the time on the same side of the origin and a 10% probability that it will spend 99.4% of the time on the same side. None of this is predicted by "intuition".

Feller also proves a change of sign theorem (see [8]) which relates to the probability that the path will cross the axis ie $S_{n-1}$and $S_{n+1}$are of opposite signs. Because this means that $S_n = 0$ n must be even. The theorem is as follows:

$$\text{The probability } \xi_{r,2n+1} \text{ that up to time } 2n+1 \text{ there occur exactly } r \text{ changes of sign equals } 2\, p_{2n+1,2r+1} \text{ ie } \xi_{r,2n+1} = 2P\{S_{2n+1} = 2r+1\} \tag{1.27}$$

Recall that $p_{n,r} = \frac{\binom{n}{\frac{n+r}{2}}}{2^n}$ where the binomial factor is interpreted as zero unless $\frac{n+r}{2}$ is an integer between 0 and n inclusive.

The probabilities of exactly r changes in 99 trials are set out below:

r	$p_{99,r}$
0	0.1592
1	0.1529
2	0.1412
3	0.1252
4	0.1066
5	0.08725
6	0.06856
7	0.05172
8	0.03745
9	0.02603
10	0.01735
11	0.01109
12	0.006799
13	0.003993

Here $p_{2n+1,2r+1} = \frac{\binom{2n+1}{\frac{2n+2r+2}{2}}}{2^{2n+1}} = \frac{\binom{2n+1}{n+r+1}}{2^{2n+1}}$ so from (1.27) $\xi_{r,2.49+1} = 2\,p_{99,r} = \frac{\binom{99}{50+r}}{2^{98}}$

Feller points out that "an amazing consequence" of (1.27) is that "the probability $\xi_{r,n}$ of r changes of sign in n trials decreases with r" (see [9]) ie

$$\xi_{0,n} \geq \xi_{1,n} > \xi_{2,n} > \ldots \tag{1.28}$$

Regardless of the number of tosses, the event that the lead never changes is more probable any preassigned number of changes. Once again, "intuition" is way off the money in predicting this.

General formula for a biased random walk

Unfair games can be modelled by a biased random walk where we define:

$$S_n = S_0 + X_1 + X_2 + \ldots + X_n \tag{1.29}$$

where $P(X_i = 1) = p$ and $P(X_i = -1) = 1 - p = q$, $p \neq q$.

What we are after is a formula for the probability that a gambler who starts with \$k will make \$A before losing \$B. This is the classic "gambler's ruin" situation. One way to fund this probability is to develop a recurrence relation and then find a solution. It is perhaps easier to start with an unbiased situation where p = q = $\frac{1}{2}$. Thus if f(k) = P($S_t = A \mid S_0 = k$) where - B ≤ k ≤ A we can develop a recurrence relation by considering what happens after one round of the game. Here t = min{ n ≥ 0: $S_n = A$ or $S_n = -B$} . After one round the gambler's wealth has either increased by \$1 with probability $\frac{1}{2}$ or decreased by \$1 with probability $\frac{1}{2}$. This suggests a recurrence relation of the form:

$$f(k) = \frac{1}{2} f(k-1) + \frac{1}{2} f(k+1) \text{ for } -B < k < A \tag{1.30}$$

There are boundary conditions as follows: f(A) = 1 and f(-B) = 0. These make sense because if you start with \$A you are certain to get to \$A before you get to -\$B, but if you "start" with -\$B you can never get to \$A before you get to -\$B.

One way to solve the recurrence equation (1.30) is to use a finite difference operator Δ defined as follows:

$$\Delta f(k-1) = f(k) - f(k-1) \tag{1.31}$$

Using this definition we derive the second difference (which is like a second derivative except that we are working with non-negative integers here) as follows:

Δ^2 f(k-1) = Δ(Δf(k-1)) = Δ (f(k)-f(k-1)) = Δf(k) - Δf(k-1)

= f(k+1)-f(k) - [f(k) - f(k-1)] = f(k+1) + f(k-1) -2 f(k)

= 2f(k) - 2f(k) using (1.30)

= 0

Thus Δ^2 f(k-1) = 0 and analogously with calculus you would guess a solution of the form f(k) = α k + β since when you "differentiate" it twice you get zero. Running with this informed guess we get the following using the two boundary probabilities:

f(-B) = -Bα + β = 0

f(A) = $A\alpha + \beta = 1$

Solving we get (A+B)α = 1 so $\alpha = \frac{1}{A+B}$

Hence f(k) = $\frac{k}{A+B} + \beta$ and using the boundary condition f(A) = 1 we see that $1 = \frac{A}{A+B} + \beta$ so that $\beta = \frac{B}{A+B}$

Thus f(k) = $\frac{k}{A+B} + \frac{B}{A+B}$ and so f(0) = $\frac{B}{A+B}$ and this is the desired probability of getting to $A before losing $B.

Returning to problem of a biased coin (see (1.29)) we simply mimic (1.30) to get:

$$f(k) = p\,f(k+1) + q\,f(k-1) \tag{1.32}$$

So we can get difference operator approach to work we can rearrange (1.32) by using the fact that p + q = 1:

(p + q) f(k) = p f(k+1) + q f(k - 1)

0 = p {f(k + 1) - f(k)} - q { f(k) - f(k-1)}

$$\text{Hence } \Delta f(k) = \frac{q}{p}\Delta f(k-1) \tag{1.33}$$

It follows from *a* simple inductive argument that $\Delta f(k+j) = \left(\frac{q}{p}\right)^j \Delta f(k)$ (1.34)

This can be seen as follows:

$\Delta f(k+1) = \left(\frac{q}{p}\right)\Delta f(k)$ ... the base case

Assume the formula holds for any j ie $\Delta f(k+j) = \left(\frac{q}{p}\right)^j \Delta f(k)$ (note that k is fixed)

$\Delta f(k+j+1) = \left(\frac{q}{p}\right)\Delta f(k+j) = \left(\frac{q}{p}\right)\left(\frac{q}{p}\right)^j \Delta f(k) = \left(\frac{q}{p}\right)^{j+1}\Delta f(k)$, establishing the proposition by induction.

There are various ways to solve (1.34) which are covered in the theory of finite difference equations. This is not the place to develop that general theory in detail but it is possible to make an "inspired" guess at the general form of the solution and then see if it makes sense. Thus we try the following:

$$f(k) = \alpha\left(\frac{q}{p}\right)^k + \beta \tag{1.35}$$

Now we know from (1.33) that f(k) must satisfy $\Delta f(k) = \frac{q}{p}\Delta f(k-1)$ and we can see if this is true for our guess.

$\Delta f(k) = f(k+1) - f(k) = \alpha\left(\frac{q}{p}\right)^{k+1} + \beta - \left(\alpha\left(\frac{q}{p}\right)^k + \beta\right) = \alpha\left\{\left(\frac{q}{p}\right)^{k+1} -\right.$

$\left(\frac{q}{p}\right)^{k}\Big\}$ = $\alpha\left(\frac{q}{p}\right)^{k}\{\frac{q}{p}-1\}$

Now $\frac{q}{p}\Delta f(k-1)$ = $\frac{q}{p}\{f(k) - f(k-1)\}$ = $\frac{q}{p}\{\alpha\left(\frac{q}{p}\right)^{k} + \beta - (\alpha\left(\frac{q}{p}\right)^{k-1} + \beta)\}$ = $\left(\frac{q}{p}\right)^{k-1}\left(\frac{q}{p}\right)\alpha\{\frac{q}{p}-1\} = \alpha\left(\frac{q}{p}\right)^{k}\{\frac{q}{p}-1\} = \Delta f(k)$

So our inspired guess looks good. The coefficients α and β are determined as follows:

f(-B) = 0 = $\alpha\left(\frac{q}{p}\right)^{-B} + \beta$

f(A)=1 = $\alpha\left(\frac{q}{p}\right)^{A} + \beta$

Hence $1 = \alpha\,\{\left(\frac{q}{p}\right)^{A} - \left(\frac{q}{p}\right)^{-B}\} = \alpha\left(\frac{q}{p}\right)^{-B}\left(\left(\frac{q}{p}\right)^{A+B}-1\right)$ so $\alpha = \frac{\left(\frac{q}{p}\right)^{B}}{\left(\frac{q}{p}\right)^{A+B}-1}$

From f(-B) = 0 it follows that $\beta = -\alpha\left(\frac{q}{p}\right)^{-B} = \frac{-\left(\frac{q}{p}\right)^{B}\left(\frac{q}{p}\right)^{-B}}{\left(\frac{q}{p}\right)^{A+B}-1} = \frac{-1}{\left(\frac{q}{p}\right)^{A+B}-1}$

Finally, $f(k) = \frac{\left(\frac{q}{p}\right)^{B+k}-1}{\left(\frac{q}{p}\right)^{A+B}-1}$ (1.36)

The required probability that the path will hit \$A before – \$B is $f(0) = \frac{\left(\frac{q}{p}\right)^{B}-1}{\left(\frac{q}{p}\right)^{A+B}-1}$ (1.37)

If you don't like or follow this derivation there is an alternative that might appeal. One can write f(k) as the sum of suitable differences as follows since terms cancel (noting that f(-B) = 0)):

f(k) = f(-B+1) - f(-B) + f(-B+2) - f(-B+1) + f(-B+3) - f(-B+2) +...+ f(k) - f(k-1)

= Δf(-B) + Δf(-B+1) + Δf(-B+2) +...+ Δf(k-1)

$= \sum_{j=0}^{k+B-1}\Delta f(-B+j)$

$= \gamma\sum_{j=0}^{k+B-1}\left(\frac{q}{p}\right)^{j}$ where $\gamma = \Delta f(-B)$ and we use (1.34)

$$= \gamma \frac{\left(\frac{q}{p}\right)^{B+k} - 1}{\left(\frac{q}{p}\right) - 1}$$

Because f(A) = 1 we have that $1 = \gamma \frac{\left(\frac{q}{p}\right)^{B+A} - 1}{\left(\frac{q}{p}\right) - 1}$ so that $\gamma = \frac{\left(\frac{q}{p}\right) - 1}{\left(\frac{q}{p}\right)^{A+B} - 1}$.

Finally, $f(k) = \frac{\left(\frac{q}{p}\right) - 1}{\left(\frac{q}{p}\right)^{A+B} - 1} \frac{\left(\frac{q}{p}\right)^{B+k} - 1}{\left(\frac{q}{p}\right) - 1} = \frac{\left(\frac{q}{p}\right)^{B+k} - 1}{\left(\frac{q}{p}\right)^{A+B} - 1}$ and we get f(0) as before.

So what is the point of all this?

The point is very simple. These complex probabilities inevitably are at odds with your "informed" intuition. In fact your intuition is not worth the paper it is not written on!! The finance world is inhabited with some people who understand this. They are people who are paid to gamble with other people's money. If you don't feel like a schmuck yet, the table that is coming up may finally convince you. The table below sets out the probabilities of winning \$100 before losing \$100 based on probabilities that are allegedly typical of the world's casinos (see [10]). Formula (1.37) drives the calculations in the table. The table assumes a constant bet of \$1 per round.

Chance to win on one round	0.500	0.495	0.490	0.480	0.470
Chance to win \$100	0.500	0.1191	0.0179	0.0003	6×10^{-6}

Now think about this: in a game with a 0.47 chance of winning on each bet how much more likely are you to win \$100 by betting \$100 on a single round than playing just \$1 per round? Do you even think the big single bet gives you a better chance? If you have a 0.47 chance of winning \$100 on one bet of \$100 this compares with a probability of 6×10^{-6} by the \$1 per round approach. Now $\frac{0.47}{6 \times 10^{-6}} = 78\,333.3$ so you are over 78000 times better off placing the big bet. It is difficult to see this sort of result "intuitively" because the relationships are essentially non-linear and as a far as I can tell, "intuition" works best with linear relationships.

The investment industry prattles on about "dollar cost averaging" whereby you drip feed small amounts into managed funds. This is a marketing spiel designed to get assets into funds so that the asset managers can feed off a bigger fee base. Simple as that. The way it is sold to investors completely lacks any rigorous intellectual basis.

The biased coin example demonstrates that the lumpy bet can be better than lots of small bets so the investment marketing spinmeisters would have to intellectually get around that. The reality is that they generally don't even know it is an issue and, even if they do, they will produce bull market data which will "prove" that dollar cost investing is effective. At the end of the day dollar cost averaging into a bull market is not a terrible thing. Drip feeding money into a bear market, however, is not a brilliant strategy and there lay the problem - your inspired financial adviser has no more idea of the future performance of markets than you do. So how do you know right now what sort of market you are in and how long will it last? No-one knows and if they did they wouldn't be telling you!

In short, such marketing spiels have much more to do with behavioural issues than they do with any analytical basis. The marketers within these institutions know that investors are generally greedy, short term thinkers and they cobble together products to match their assessment of that behaviour. When markets turn from greed to fear they simply wheel out the fear products. The deluded punter will get charged too much in both cycles of course.

In the end you are on your own and that frightens many people who still believe in fairies with magic wands. If you think the investment industry cares about your interests in any meaningful sense then you truly do deserve the epithet "schmuck".

A

Appendix - proof of the hypergeometric series formula

The hypergeometric series formula has the following form for positive integers a, b and n:

$$\binom{a}{0}\binom{b}{n}+\binom{a}{1}\binom{b}{n-1}+\ldots+\binom{a}{n-1}\binom{b}{1}+\binom{a}{n}\binom{b}{0} = \binom{a+b}{n} \qquad (A.1)$$

There are various ways to prove (A.1). Once can use induction (one of Feller's problems is along those lines - see [11]), however, the most straightforward approach is to use appropriate binomial expansions and equate coefficients. Thus all you need to do is equate the coefficients of x^n on both sides of $(1+x)^a(1+x)^b = (1+x)^{a+b}$. Expressing each side in terms of its binomial expansions we have:

$$\left(\sum_{k=0}^{a}\binom{a}{k}x^k\right)\left(\sum_{j=0}^{b}\binom{b}{j}x^j\right) = \sum_{j=0}^{a+b}\binom{a+b}{j}x^j$$

ie $\sum_{k=0}^{a}\sum_{j=0}^{b}\binom{a}{k}\binom{b}{j}x^{k+j} = \sum_{j=0}^{a+b}\binom{a+b}{j}x^j$

The coefficient of x^n on the RHS is $\binom{a+b}{n}$ while on the LHS it is obtained by taking the pairs (k,j) in the sum of products $\binom{a}{k}\binom{b}{j}$ where k = s and j = n - s for $0 \le s \le n$. In other words:

$$\binom{a}{0}\binom{b}{n} + \binom{a}{1}\binom{b}{n-1} + ... + \binom{a}{n-1}\binom{b}{1} + \binom{a}{n}\binom{b}{0} = \binom{a+b}{n}$$

There is another way of proving the formula which draws upon more detailed combinatorial arguments that are instructive in themselves. Start with a population of n elements of which n_1 are white and n_2= n - n_1 are black. A subset of r objects is chosen at random. To find the probability p_k that this group will contain exactly k white elements will involve the hypergeometric formula. Note first that $0 \le k \le \min\{n_1, r\}$. To find the required probability you need to work out the proportion of favourable cases which comprises k white and r - k black elements.

The number of distinct ways of choosing the white objects is $\binom{n_1}{k}$

The number of distinct ways of choosing the black objects is $\binom{n-n_1}{r-k}$

Because any of the white elements can be combined with any choice of the black elements the number of favourable cases is $\binom{n_1}{k}\binom{n-n_1}{r-k}$. Since there are $\binom{n}{r}$ ways of choosing r elements from n the required probability is:

$$p_k = \frac{\binom{n_1}{k}\binom{n-n_1}{r-k}}{\binom{n}{r}} \qquad \text{(A.2)}$$

Formula (A.2) can be recast as follows:

$$p_k = \frac{\binom{n_1}{k}\binom{n-n_1}{r-k}}{\binom{n}{r}} = p_k = \frac{(n_1)!\,(n-n_1)!\,(n-r)!\,r!}{(n-k)!\,k!\,(n-n_1-r+k)!\,(r-k)!\,n!}$$

$$= \frac{\binom{r}{k}\binom{n-r}{n_1-k}}{\binom{n}{n_1}} \qquad \text{(A.3)}$$

Note here that p_k= 0 if k > r or k > n_1 since $\binom{x}{y}$ = 0 if y > x.

Because the p_k are probabilities they must sum to 1so that:

$$p_0 + p_1 + p_2 + ... = 1 \qquad \text{(A.4)}$$

Using (A.3) and (A.4) we get:

$$\frac{\binom{r}{0}\binom{n-r}{n_1}+\binom{r}{1}\binom{n-r}{n_1-1}+\binom{r}{2}\binom{n-r}{n_1-2}+\dots+\binom{r}{n_1}\binom{n-r}{0}}{\binom{n}{n_1}}=1$$

So $\binom{r}{0}\binom{n-r}{n_1}+\binom{r}{1}\binom{n-r}{n_1-1}+\dots+\binom{r}{n_1}\binom{n-r}{0}=\binom{n}{n_1}$

Let a = r and b = n - r so that n = a + b then:

$\binom{a}{0}\binom{b}{n_1}+\binom{a}{1}\binom{b}{n_1-1}+\dots+\binom{a}{n_1}\binom{b}{0}=\binom{a+b}{n_1}$ which has the same form as (A.1)

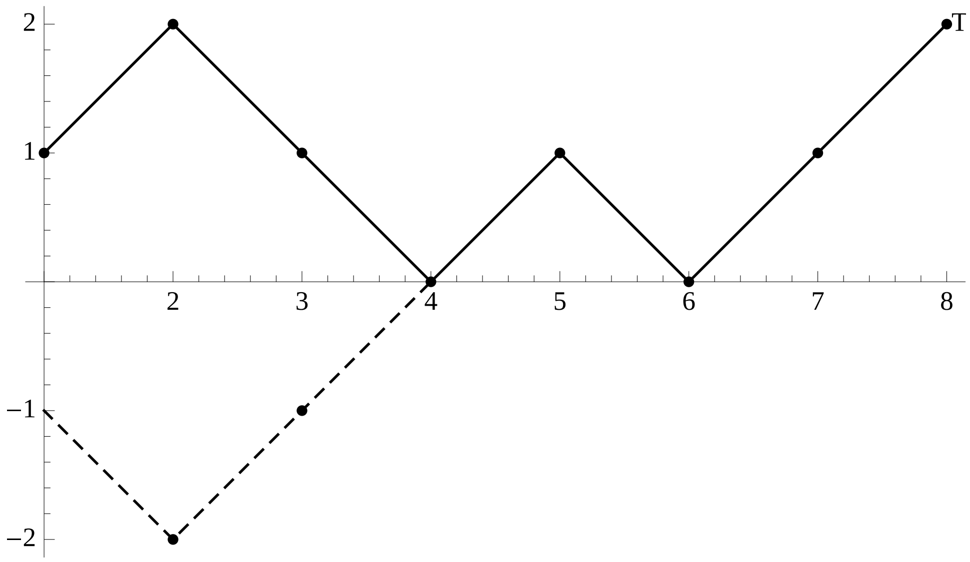
B

Appendix - the reflection principle

The reflection principle is a technique developed to calculate the number of paths in a lattice. It is used in the proof of the so-called "Ballot Problem". The Ballot Problem can be stated as follows:

Given positive integers $a, \ b$ with $a > b$,
find the number of lattice paths starting at the origin and consisting of a upsteps (B.1)
$(, 1)$ and b downsteps $(1, -1)$ such that no step ends on the $x-$axis.

The Ballot Problem is based on the concept of votes for candidate A as upsteps (1,1) and votes for candidate B are downsteps (1,-1). A ballot permutation (or path) that satisfies the ballot problem conditions is called "good" while those that don't are called "bad". The reflection principle relies upon a geometrical insight which is based on the following type of setup:



The *Mathematica* code to generate this graph is set out below:

```
data = {{1,1},{2,2},{3,1},{4,0},{5,1},{6,0},{7,1},{8,2}};
data2 = {{1,-1},{2,-2},{3,-1},{4,0}};
Graphics[{Line[data], {AbsolutePointSize[4], Point[{data}]},
    {AbsoluteDashing[{4}],Line[{data2}]}, {AbsolutePointSize[4], Point[{{2,-2},{3,-1}}]}},
    Axes→True,Epilog→{ Text["T",{8.1,2}]}]
```

Now T is the terminal point of the path with coordinates (a + b, a - b). Why? Because if there are a upsteps and b downsteps you must have advanced a + b along the horizontal (x)-axis and a

- b along the vertical (t) axis. Every good path must start at (1,1) ie from an upstep and must never touch the x-axis so that the number of good paths equals the number of paths from (1,1) to T that never touch the x - axis. Now by reflecting the paths above the x-axis which do in fact touch the x-axis you set up a one to one correspondence between that set of paths from (1,1) to T and all paths from (1,-1) to T. Note that the reflection is about the point

The total number of paths from (1,1) to T is $\binom{a+b-1}{a-1}$ (see (1.4) while noting that we are starting 1 step away from the origin). From this number we subtract the number of paths arising from the reflected set, namely, $\binom{a+b-1}{b-1}$, symmetrical with the previous case.

Thus the total number of "good" paths from $(1, 1)$ to T is $\binom{a+b-1}{a-1} -$

$$\binom{a+b-1}{b-1} = \frac{a-b}{a+b}\binom{a+b}{a} \qquad \text{(B.2)}$$

Citations

1. William Feller, "Introduction to Probability Theory and its Applications", Third Edition, Volume 1, Wiley, 1968, p.79
2. [1] p.84
3. [1] footnote 12 page 82
4. [1] page 96
5. [1] pages 76-77
6. [1] footnote 11, page 78
7. [1] pages 82-3
8. [1] pages 84 - 5
9. [1] page 85
10. (11) J. Michael Steele, "Stochastic Calculus and Financial Applications", Springer, 2001, page 7
11. [1] Problem 12.9 page 64

History:
Created March 2011
16 May 2013
Typo and graph on page 6 corrected thanks to a a vigilant reader