Why Ramsey's Theorem makes sense

- a brief explanation

What is Ramsey's Theorem?

The general idea

Ramsey's Theory in its most general form requires some preliminary technical definitions, but for the purposes of getting the general idea across we can avoid that overhead. Ramsey's Theorem is a species of theorem which asserts that every system of a certain type possesses a large subsystem which a higher degree of organisation than the original system. Erdos and Szekeres proved that any sequence of length n^2+1 contains a monotone subsequence of length n+1. The infinite version of Ramsey's Theory Implies that every infinite sequence contains an infinite monotone subsequence. The Bolzano-Weierstrass Theorem which states that every bounded sequence of complex numbers contains a convergent subsequence is yet another example of Ramsey's Theorem at work.

The paradigm case for understanding Ramsey's Theorem is the following proposition:

In any collection of six people either three of them mutually know each other or three of them mutually do not know each other.

Let's call this the "basic Ramsey result".

How to prove the basic Ramsey result

It is assumed that knowing someone is symmetric ie if A knows B then B knows A. Knowing is not assumed to be transitive ie if A knows B and B knows C it is not assumed that A thereby knows C. A may or may not know C.

Choose one of the six people, A say, and consider A's relation of knowing to the remaining five people B, C, D, E and F. If A knows only B, say, then there are four people A does not know. If A knows B and C then there are three people A does not know. If A knows B, C and D then there only two people that A does not know. Going in the other direction, if A does not know B, then there are four people A knows and if A does not know B and C there are three people that A knows. If A does not know B, C and D then there are only two people A knows. Thus A must either know at least three of the group of five or not know at least three of them.

Now suppose that A knows three of the five, say D, E and F. If some pair of these three know each other, E and F for example, then A, E and F are three people who mutually know one another. On the other hand if no pair among the three know one another it follows that we have a group of three people who do not mutually know one another. Either way we have a group of three with the required property.

Now go in the other direction ie suppose that A does not know three of the five, say B, C and D. If some pair of B, C and D do not know each other (C and D for example) then A, C and D do not mutually know each other and we have our group of three. On the other hand if there is no pair among the three who don't know one another, all three must know one another mutually and we have a group of three with the required property.

You can see why Ramsey's original proof was titled "*On a problem of formal logic*". In formal logic courses you prove the following things:

(1) symmetry and transitivity together imply reflexivity
(2) asymmetry implies irreflexivity
(3) intransitivity implies irreflexivity
(4) transitivity and irreflexivity together imply asymmetry

See *W V Quine, "Methods of Logic", Revised Edition, Routledge & Kegan Paul,1972, pages 160 -161*

The theorem Ramsey proved in his paper read as follows:

"Let Γ be an infinite class, and μ and r positive integers, and let all those sub-classes of Γ which have exactly r members, or, as we may say, let all r-combinations of the members of Γ be divided in any manner into μ mutually exclusive classes $C_i(i = 1, 2, ..., \mu)$, so that every $r-$combination is a member of one and only one C_i; then, assuming the Axiom of Selections, Γ must contain an infinite sub-class Δ such that all the r-combinations of the members of Δ belong to the same C_i" (See *Ronald L Graham, Bruce L Rothschild and Joel H Spencer, "Ramsey Theory", Second Edition, John Wiley & Sons, 1990, page 19*)

Ramsey starts with $\mu = 2$ and proves the theorem for all values of r by induction.

In 1927 van der Waerden published a proof (using a form of double induction) of the following remarkable result:

If the positive integers are partitioned into two classes then at least one of the classes must contain arbitrarily long arithmetic progressions.

Density and compactness theorems also reflect Ramsey theorem thinking which is fundamental to advanced combinatorial theory.

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